COMPACT WEIGHTED COMPOSITION OPERATORS
ON BANACH LATTICES

WILLIAM FELDMAN

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ABSTRACT. A characterization of compact (and \( M \)-weakly compact) weighted composition operators on real and complex Banach lattices which can be appropriately realized as function spaces is provided.

The compact weighted composition operators on \( C^b(X) \), all bounded continuous complex valued functions on \( X \) with the supremum norm, have been characterized in the case of compact \( X \) by Kamowitz [4], and for completely regular \( X \) by Singh and Summers [6]. We recall that an operator \( T \) on \( C^b(X) \) is called a weighted composition operator if there exists an \( r \) on \( C^b(X) \) and a continuous map \( \phi \) from \( X \) to \( X \) so that for each \( f \) in \( C^b(X) \), we have \( Tf(x) = r(x)f(\phi(x)) \).

In this note we will characterize a class of compact weighted composition operators (and compact composition operators) on more general function spaces, which we will call Banach \( F \)-lattices. These include the \( L^p \)-spaces and the Banach lattices, both real and complex, with quasi-interior points (more generally, topological order partitions). We also demonstrate that these operators are equivalent to the \( M \)-weakly compact weighted composition operators.

We will call a complex Banach space \( E \) a Banach \( F \)-lattice if there exists a real Banach lattice \( G \), which can be identified with equivalence classes of real valued functions on a completely regular space \( X \), and \( E = G + iG \) can be identified with equivalence classes of complex valued functions on \( X \) satisfying the following conditions:

(i) If \( f \) is equivalent to \( g \), then \( \{ x \in X : f(x) = g(x) \} \) is dense.
(ii) Each continuous complex valued bounded function on \( X \) (or function with compact support in the case of a locally compact \( X \) ) represents an element in \( E \).
(iii) The vector space operations on \( E \) and the lattice operations on \( G \) correspond to the pointwise defined operations.

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(iv) For each $f$ in $E$, the function $|f|(x) = |f(x)|$ is a well-defined element in $E$ and $\|f\| = \||f|\|$.

(v) Given $h \geq 0$, a bounded continuous real valued function on $X$ and $f$ in $E$, the pointwise product $hf$ is a well-defined element of $E$.

The $L^p$ spaces are examples of Banach $F$-lattices. We will verify that Banach lattices (complex as well as real) with locally compact representation spaces are also Banach $F$-lattices.

We recall (see [2] or [5]) that a real Banach lattice $G$ is said to have a locally compact representation space $X$ if the space $C_k(X, \mathbb{R})$, all real-valued continuous functions on $X$ with compact support, can be identified with a dense (order) ideal in $G$. In fact, $G$ has a locally compact representation space if and only if $E$ contains a topological order partition (see [2]). We will say that a complex Banach lattice (in the sense of Schaefer [5]) has a locally compact representation space $X$ if $E = G + iG$ and $G$ has a locally compact representation space $X$.

Given that $E$ has a locally compact representation space $X$, the real Banach lattice $G$ can be identified with continuous extended real-valued functions on $X$ (i.e. range the two point compactification of $\mathbb{R}$), each finite on a dense subset. Thus $E$ can be identified with continuous functions from $X$ to the one point compactification of $\mathbb{C}$, each finite on a dense subset of $X$. Now for $f$ in $E$, a complex valued function $g$ will be said to be equivalent to $f$ if $\{x \in X : f(x) = g(x)\}$ is a dense open subset of $X$. We will now establish that $E$ is a Banach $F$-lattice via the identification of each $f$ in $E$ with this equivalence class of complex functions on $X$.

**Lemma.** Let $E$ be a complex Banach lattice with locally compact representation space $X$. Then $E$ is a Banach $F$-lattice.

**Proof.** We note that condition (iv) follows from the definition of a complex Banach lattice (see [5, p. 137]) and thus we need only verify that (v) is satisfied. Let $h \geq 0$ be a bounded continuous real-valued function on $X$ and let $f$ be in $E$. Then $f = f_1 + if_2$ as a function on $X$ where $f_1$ and $f_2$ are real-valued. We view $f_1$ as a continuous function from $X$ to the two point compactification of $\mathbb{R}$ and choose an increasing net $(f_{\alpha})$ of functions in $C_k(X, \mathbb{R})$ converging up to $f_1$ in norm and pointwise for all $x$. Now $(hf_{\alpha})$ is a Cauchy net since $|hf_{\alpha} - hf_{\beta}| \leq \|h\|_{\infty} |f_{\alpha} - f_{\beta}|$ so that $\|hf_{\alpha} - hf_{\beta}\| \leq \|h\|_{\infty} \|f_{\alpha} - f_{\beta}\|$. Hence $(hf_{\alpha})$ converges to some element $g$ in $G$ which we identify with an extended real valued function. If there exists an $x_0$ in $X$ such that $g(x_0) > hf_1(x_0)$ then $g$ exceeds $hf_1$ on a compact neighborhood $N$ of $x_0$. Let $k \geq 0$ be in $C_k(X, \mathbb{R})$ such that $k$ vanishes off of $N$, and $(g - hf_1(x)) \geq k(x)$ for all $x$. Now $(g - hf_1)$ exceeds $k$ for all $x$ and $\alpha$, so that $\|g - hf_{\alpha}\| \geq \|k\|$, a contradiction. Hence $g \leq hf_1$. If there exists an $x_0$ such that $g(x_0) < hf_1(x_0)$, then $g(x_0) < hf_{\alpha}(x_0)$ for some $\alpha$. Again $g < hf_{\alpha}$ on a neighborhood $N$ of $x_0$, and we choose a function $k \geq 0$ with $k \leq (hf_{\alpha} - g)$. Thus $\|k\| \leq \|hf_{\alpha} - g\|$.
for all $\alpha > \hat{\alpha}$, and we conclude that $hf_1 = g$. A similar argument applies to $f_2$.

Given $E$ and $F$ Banach $F$-lattices identified with equivalence classes of complex valued functions on $X$ and $Y$ respectively, we will call a bounded operator $\Phi$ from $E$ to $F$ a composition operator if there exists a continuous function $\phi$ from $Y$ to $X$ so that

$$\Phi(f)(y) = f(\phi(y))$$

for each $f$ in $E$ and $y$ in $Y$. A bounded operator $T$ will be called a weighted composition operator if there exists a continuous complex valued function $r$ on $Y$ and a composition operator $\Phi$ that

$$T(f)(y) = r(y)\Phi f(y) = r(y)f(\phi(y))$$

We will often write $T = r(f \circ \phi)$.

A weighted composition operator $T$ will be said to satisfy condition (*) if there exists a $\delta \geq 0$ so that for each $x$ in $\phi(Y)$ and neighborhood $N$ of $x$ there is an element $f$ in $E$ with $\|f\| = 1$ equivalent to a function vanishing outside of $N$, and $\|\Phi(f)\| \geq \delta$.

Although many operators will satisfy condition (*), for example, translations or shifts, we cite an elementary example where this condition fails. On $L^1[0,1]$, let $\Phi(f) = f(\sqrt{x})$ ($\phi(x) = \sqrt{x}$). At $x = 0$ condition (*) is not satisfied.

**Theorem 1.** Let $E$ and $F$ be Banach $F$-lattices identified with equivalence classes of functions on $X$ and $Y$ respectively, and $T$ a weighted composition operator from $E$ to $F$. Let $r$ be in $C(Y)$, and $\phi$ a continuous map from $Y$ to $X$, so that $Tf = r(f \circ \phi)$ for each $f$ in $E$. If $T$ satisfies condition (*) and for each $\varepsilon > 0$, there is a positive $\delta$ less than $\varepsilon$, so that $\phi\{y: |r(y)| \leq \delta\}$ is disjoint from $\phi\{y: |r(y)| \geq \varepsilon\}$, then the following statements are equivalent:

(i) $T$ is a compact operator.

(ii) For each $\varepsilon > 0$, the set $\phi\{y: |r(y)| > \varepsilon\}$ is finite.

**Proof.** We first assume that condition (ii) is not satisfied. Thus there exists an $\varepsilon > 0$ and a sequence $(a_i)$ of distinct points in the image of $\{y: |r(y)| \geq \varepsilon\}$ under the map $\phi$. We construct a collection of disjoint closed neighborhoods for an infinite collection of points in $(a_i)$ as follows. Choosing a subsequence if necessary, we assume that no subsequence of $(a_i)$ converges to a point in $(a_i)$. Now let $N_1$ be a neighborhood of $a_1$ such that $(a_i) - N_1$ is an infinite set and $N_1$ is disjoint from the set $F = \phi\{y: |r(y)| < \varepsilon\}$. Let $a_2$ be a point in $(a_i) - N_1$ and choose a closed neighborhood $N_2$ of $a_2$ disjoint from $N_1$ and $F$ such that $(a_i) - (N_1 \cup N_2)$ is an infinite set. Continuing inductively, let $N_i$ be a neighborhood of a point $a_i$ where $a_i$ is in the infinite set $(a_i) - \bigcup_{k<i} N_k$, disjoint from $\bigcup_{k<i} N_k$ and $F$ such that $\{a_i\} - \bigcup_{k \leq i} N_k$ is infinite. Now condition (*) implies that for each $i$ there exists an $f_i$ in $E$ with $f_i$
equal to zero on the complement of $N_i$, $\|f_i\| = 1$, and $\|\Phi(f_i)\| \geq \delta$. Now for $j \neq k$,
\[\|Tf_j - Tf_k\| = \|r(f_j \circ \phi) - r(f_k \circ \phi)\| \geq \|r(f_j \circ \phi)\| \geq \delta \|\Phi(f_j)\| \geq \delta \delta\]
since $f_j$ and $f_k$ have disjoint support and $|r|$ exceeds $\delta$ on the points where $(f_j \circ \phi)$ is nonzero. Thus the sequence $(Tf_j)$ has no convergent subsequence.

Assume that (ii) is satisfied. Given $\varepsilon > 0$, let $H$ be a positive continuous real-valued function on $Y$ which is one on \(\{y : |r(y)| > \varepsilon\}\) and zero on \(\{y : |r(y)| < \varepsilon/2\}\), and bounded by one, (e.g., $2/\varepsilon(|r| - 1/2) \wedge 1$). Now for $f$ in the unit ball of $E$
\[\|HTf - Tf\| < \varepsilon \|f \circ \phi\| \leq \varepsilon \|\Phi\|,
\]
where $\|\Phi\|$ is the operator norm of $\Phi$. We will complete the proof by showing that $HT$ is a finite rank operator (i.e., $T$ is the limit of finite rank operators). Let $\phi(y : |r(y)| \geq \varepsilon/2)$ be denoted by $\{a_i : i = 1, 2, \ldots, n\}$ and for each $i$ choose a continuous bounded real-valued function $L_i$ so that $L_i(a_i) = 1$ and $L_i(a_j) = 0$ for $i \neq j$. Now for each $f$ and $i$, there exists a constant $\alpha_i(f)$ so that
\[H(y)r(y)(L_i \circ \phi)(y)(f \circ \phi)(y) = \alpha_i(f)H(y)r(y)(L_i \circ \phi)(y)
\]
for all $y$ in $Y$. Hence $HTf = \sum_{i=1}^{n} \alpha_i(f)Hr(L_i \circ \phi)$.

**Corollary.** Let $E$ and $F$ be Banach $F$-lattices of functions on $X$ and $Y$ respectively. Let $\Phi$ be a composition operator such that $\Phi(f) = f \circ \phi$ for $\phi$ a continuous map from $Y$ to $X$ and $f$ in $E$. If $\Phi$ satisfies condition $(\ast)$, then $\Phi$ is compact if and only if the image of $Y$ under $\phi$ is finite.

In the special case that $E$ and $F$ are spaces of all continuous bounded functions on a completely regular space with the sup-norm topology, the $(\ast)$ assumption and the separation assumption are not necessary in the previous results. Thus we provide an alternative proof for the result established by Singh and Summers in [6], and, in a slightly different form, by Kamowitz for compact $X$ in [4].

**Theorem 2.** Let $C^b(X)$ and $C^b(Y)$ be the spaces of all continuous complex valued bounded functions on $X$ and $Y$ respectively with the sup-norm topology. Let $\phi$ be a continuous map from $Y$ to $X$ and $r$ a continuous bounded complex valued function on $Y$.

(a) The composition operator $\Phi$ defined by $\Phi(f) = f \circ \phi$ for each $f$ in $C^b(X)$ is compact if and only if the image of $Y$ under $\phi$ is finite.

(b) The weighted composition operator $T$ defined by $Tf = r(f \circ \phi)$ for each $f$ in $C^b(X)$ is compact if and only if for each $\varepsilon > 0$, the image under $\phi$ of \(\{y : |r(y)| \geq \varepsilon\}\) is finite.

**Proof.** We need only prove part (b) and proceed as in the proof of Theorem 1. Assume first that there is an infinite sequence of distinct points $(a_i)$ in the
image of \( \{ y : |r(y)| \geq \varepsilon \} \) under the map \( \phi \). We choose neighborhoods \( N_i \) as in the proof of Theorem 1 without regard to the set \( F \) (i.e., without \( N_i \) disjoint from \( F \)). Then the function \( f_j \) is chosen to be one at \( a_{ij} \) and zero on the complement of \( N_i \) with \( 0 \leq f_j \leq 1 \). For \( y \) in \( Y \) with \( r(y) \geq \varepsilon \) and \( \phi(y) = a_{ij} \), we have

\[
\| T f_j - T f_k \| \geq \| r(f_j \circ \phi) - r(f_k \circ \phi) \| \geq |r(y) f_j (a_{ij})| = |r(y)| \geq \varepsilon.
\]

The proof of the converse is the same as the proof in Theorem 1.

We recall that (e.g., see [1, p. 313]) that an operator from \( E \) to \( F \) is called \( M \)-weakly compact if for every disjoint sequence \( (f_n) \) in \( E \) bounded in norm, the sequence \( (\| T(f_n) \|) \) converges to zero \( ( f \) is disjoint from \( g \) if \( |f| \wedge |g| = 0 \)).

We can now prove the following:

**Theorem 3.** A weighted composition operator \( T \) satisfying the hypothesis of Theorem 1 is compact if and only if \( T \) is \( M \)-weakly compact.

**Proof.** We will verify that \( T \) is \( M \)-weakly compact if and only if condition (ii) of Theorem 1 is satisfied. Given that (ii) is not satisfied, the same argument as in the proof of Theorem 1 yields a bounded disjoint sequence whose image under \( T \) does not converge to zero in norm. Given that (ii) is satisfied, let \( H \) be the function in the proof of Theorem 1. We note that for \( (f_j) \) a disjoint sequence bounded in \( E \), the sequence \( (HT f_j) \) is also disjoint. Since \( \phi(\{ y : r(y) \geq \varepsilon/2 \}) \) is finite, at most finitely many of the functions \( HT f_j \) can be nonzero. Thus \( (HT f_j) \) is convergent to zero in norm. Now \( HT \) is \( M \)-weakly compact and since the \( M \)-weakly compact operators are closed (e.g., [1, p. 317]), we conclude that \( T \) is \( M \)-weakly compact.

**References**