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COMPACT WEIGHTED COMPOSITION OPERATORS ON BANACH LATTICES

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ABSTRACT. A characterization of compact (and M-weakly compact) weighted composition operators on real and complex Banach lattices which can be appropriately realized as function spaces is provided.

The compact weighted composition operators on $C^b(X)$, all bounded continuous complex valued functions on X with the supremum norm, have been characterized in the case of compact X by Kamowitz [4], and for completely regular X by Singh and Summers [6]. We recall that an operator T on $C^b(X)$ is called a weighted composition operator if there exists an r on $C^b(X)$ and a continuous map ϕ from X to X so that for each f in $C^b(X)$, we have $Tf(x) = r(x)f(\phi(x))$.

In this note we will characterize a class of compact weighted composition operators (and compact composition operators) on more general function spaces, which we will call Banach F-lattices. These include the L^p-spaces and the Banach lattices, both real and complex, with quasi-interior points (more generally, topological order partitions). We also demonstrate that these operators are equivalent to the M-weakly compact weighted composition operators.

We will call a complex Banach space E a *Banach F-lattice* if there exists a real Banach lattice G, which can be identified with equivalence classes of real valued functions on a completely regular space X, and $E = G + iG$ can be identified with equivalence classes of complex valued functions on X satisfying the following conditions:

(i) If f is equivalent to g, then $\{x \in X : f(x) = g(x)\}$ is dense.
(ii) Each continuous complex valued bounded function on X (or function with compact support in the case of a locally compact X) represents an element in E.
(iii) The vector space operations on E and the lattice operations on G correspond to the pointwise defined operations.

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(iv) For each f in E, the function $|f|(x) = |f(x)|$ is a well-defined element in E and $\|f\| = \| |f| \|$.

(v) Given $h \geq 0$, a bounded continuous real valued function on X and f in E, the pointwise product hf is a well-defined element of E.

The L^p spaces are examples of Banach F-lattices. We will verify that Banach lattices (complex as well as real) with locally compact representation spaces are also Banach F-lattices.

We recall (see [2] or [5]) that a real Banach lattice G is said to have a *locally compact representation space* X if the space $C_k(X, \mathbf{R})$, all real-valued continuous functions on X with compact support, can be identified with a dense (order) ideal in G. In fact, G has a locally compact representation space if and only if E contains a topological order partition (see [2]). We will say that a complex Banach lattice (in the sense of Schaefer [5]) has a *locally compact representation space* X if $E = G + iG$ and G has a locally compact representation space X.

Given that E has a locally compact representation space X, the real Banach lattice G can be identified with continuous extended real-valued functions on X (i.e. range the two point compactification of $\mathbf{R}$), each finite on a dense subset. Thus E can be identified with continuous functions from X to the one point compactification of $\mathbf{C}$, each finite on a dense subset of X. Now for f in E, a complex valued function g will be said to be equivalent to f if $\{x \in X\colon f(x) = g(x)\}$ is a dense open subset of X. We will now establish that E is a Banach F-lattice via the identification of each f in E with this equivalence class of complex functions on X.

Lemma. *Let E be a complex Banach lattice with locally compact representation space X. Then E is a Banach F-lattice.*

Proof. We note that condition (iv) follows from the definition of a complex Banach lattice (see [5, p. 137]) and thus we need only verify that (v) is satisfied. Let $h \geq 0$ be a bounded continuous real-valued function on X and let f be in E. Then $f = f_1 + if_2$ as a function on X where f_1 and f_2 are real-valued. We view f_1 as a continuous function from X to the two point compactification of $\mathbf{R}$ and choose an increasing net (f_α) of functions in $C_k(X, \mathbf{R})$ converging up to f_1 in norm and pointwise for all x. Now (hf_α) is a Cauchy net since $|hf_\alpha - hf_\beta| \leq \|h\|_\infty |f_\alpha - f_\beta|$ so that $\|hf_\alpha - hf_\beta\| \leq \|h\|_\infty \|f_\alpha - f_\beta\|$. Hence (hf_α) converges to some element g in G which we identify with an extended real valued function. If there exists an x_0 in X such that $g(x_0) > hf_1(x_0)$ then g exceeds hf_1 on a compact neighborhood N of x_0. Let $k \geq 0$ be in $C_k(X, \mathbf{R})$ such that k vanishes off of N, and $(g - hf_1(x)) \geq k(x)$ for all x. Now $(g - hf_\alpha)$ exceeds k for all x and α, so that $\|g - hf_\alpha\| \geq \|k\|$, a contradiction. Hence $g \leq hf_1$. If there exists an x_0 such that $g(x_0) < hf_1(x_0)$, then $g(x_0) < hf_{\hat{\alpha}}(x_0)$ for some $\hat{\alpha}$. Again $g < hf_{\hat{\alpha}}$ on a neighborhood N of x_0, and we choose a function $k \geq 0$ with $k < (hf_{\hat{\alpha}} - g)$. Thus $\|k\| \leq \|hf_\alpha - g\|$

for all $\alpha > \hat{\alpha}$, and we conclude that $hf_1 = g$. A similar argument applies to f_2.

Given E and F Banach F-lattices identified with equivalence classes of complex valued functions on X and Y respectively, we will call a bounded operator Φ from E to F a composition operator if there exists a continuous function ϕ from Y to X so that

$$\Phi(f)(y) = f(\phi(y))$$

for each f in E and y in Y. A bounded operator T will be called a weighted composition operator if there exists a continuous complex valued function r on Y and a composition operator Φ that

$$T(f)(y) = r(y)\Phi f(y) = r(y)f(\phi(y))$$

We will often write $T = r(f \circ \phi)$.

A weighted composition operator T will be said to satisfy condition $(*)$ if there exists a $\delta \geq 0$ so that for each x in $\phi(Y)$ and neighborhood N of x there is an element f in E with $\|f\| = 1$ equivalent to a function vanishing outside of N, and $\|\Phi(f)\| \geq \delta$.

Although many operators will satisfy condition $(*)$, for example, translations or shifts, we cite an elementary example where this condition fails. On $L^1[0, 1]$, let $\Phi(f) = f(\sqrt{x})$ $(\phi(x) = \sqrt{x})$. At $x = 0$ condition $(*)$ is not satisfied.

Theorem 1. *Let E and F be Banach F-lattices identified with equivalence classes of functions on X and Y respectively, and T a weighted composition operator from E to F. Let r be in $C(Y)$, and ϕ a continuous map from Y to X, so that $Tf = r(f \circ \phi)$ for each f in E. If T satisfies condition $(*)$ and for each $\varepsilon > 0$, there is a positive $\hat{\varepsilon}$ less than ε, so that $\overline{\phi\{y \colon |r(y)| \leq \hat{\varepsilon}\}}$ is disjoint from $\phi\{y \colon |r(y)| \geq \varepsilon\}$, then the following statements are equivalent:*

(i) *T is a compact operator.*
(ii) *For each $\varepsilon > 0$, the set $\phi\{y \colon |r(y)| > \varepsilon\}$ is finite.*

Proof. We first assume that condition (ii) is not satisfied. Thus there exists an $\varepsilon > 0$ and a sequence (a_i) of distinct points in the image of $\{y \colon |r(y)| \geq \varepsilon\}$ under the map ϕ. We construct a collection of disjoint closed neighborhoods for an infinite collection of points in $\{a_i\}$ as follows. Choosing a subsequence if necessary, we assume that no subsequence of (a_i) converges to a point in $\{a_i\}$. Now let N_1 be a neighborhood of a_1 such that $\{a_i\} - N_1$ is an infinite set and N_1 is disjoint from the set $F = \overline{\phi\{y \colon |r(y)| < \hat{\varepsilon}\}}$. Let a_{i_2} be a point in $\{a_i\} - N_1$ and choose a closed neighborhood N_2 of a_{i_2} disjoint from N_1 and F such that $\{a_i\} - (N_1 \cup N_2)$ is an infinite set. Continuing inductively, let N_j be a neighborhood of a point a_{i_j} where a_{i_j} is in the infinite set $\{a_i\} - \bigcup_{k<j} N_k$, disjoint from $\bigcup_{k<j} N_k$ and F such that $\{a_i\} - \bigcup_{k\leq j} N_k$ is infinite. Now condition $(*)$ implies that for each i there exists an f_i in E with f_i

equal to zero on the complement of N_i, $\|f_i\| = 1$, and $\|\Phi(f_i)\| \geq \delta$. Now for $j \neq k$,

$$\|Tf_j - Tf_k\| = \|r(f_j \circ \phi) - r(f_k \circ \phi)\| \geq \|r(f_j \circ \phi)\| \geq \hat{\varepsilon}\|\Phi(f_j)\| \geq \hat{\varepsilon}\delta$$

since f_j and f_k have disjoint support and $|r|$ exceeds $\hat{\varepsilon}$ on the points where $(f_j \circ \phi)$ is nonzero. Thus the sequence (Tf_i) has no convergent subsequence.

Assume that (ii) is satisfied. Given $\varepsilon > 0$, let H be a positive continuous real-valued function on Y which is one on $\{y: |r(y)| \geq \varepsilon\}$ and zero on $\{y: |r(y)| < \varepsilon/2\}$, and bounded by one, (e.g., $2/\varepsilon(|r| - \frac{1}{2}\varepsilon) \wedge \frac{1}{2}\varepsilon)^+$. Now for f in the unit ball of E

$$\|HTf - Tf\| \leq \|\varepsilon|f \circ \phi|\| \leq \varepsilon\|\Phi\|,$$

where $\|\Phi\|$ is the operator norm of Φ. We will complete the proof by showing that HT is a finite rank operator (i.e., T is the limit of finite rank operators). Let $\phi\{y: |r(y)| \geq \varepsilon/2\}$ be denoted by $\{a_i: i = 1, 2, \ldots, n\}$ and for each i choose a continuous bounded real-valued function L_i so that $L_i(a_i) = 1$ and $L_i(a_j) = 0$ for $i \neq j$. Now for each f and i, there exists a constant $\alpha_i(f)$ so that

$$H(y)r(y)(L_i \circ \phi)(y)(f \circ \phi)(y) = \alpha_i(f)H(y)r(y)(L_i \circ \phi)(y)$$

for all y in Y. Hence $HTf = \sum_{i=1}^n \alpha_i(f)Hr(L_i \circ \phi)$.

Corollary. *Let E and F be Banach F-lattices of functions on X and Y respectively. Let Φ be a composition operator such that $\Phi(f) = f \circ \phi$ for ϕ a continuous map from Y to X and f in E. If Φ satisfies condition $(*)$, then Φ is compact if and only if the image of Y under ϕ is finite.*

In the special case that E and F are spaces of all continuous bounded functions on a completely regular space with the sup-norm topology, the $(*)$ assumption and the separation assumption are not necessary in the previous results. Thus we provide an alternative proof for the result established by Singh and Summers in [6], and, in a slightly different form, by Kamowitz for compact X in [4].

Theorem 2. *Let $C^b(X)$ and $C^b(Y)$ be the spaces of all continuous complex valued bounded functions on X and Y respectively with the sup-norm topology. Let ϕ be a continuous map from Y to X and r a continuous bounded complex valued function on Y.*

(a) *The composition operator Φ defined by $\Phi(f) = f \circ \phi$ for each f in $C^b(X)$ is compact if and only if the image of Y under ϕ is finite.*

(b) *The weighted composition operator T defined by $Tf = r(f \circ \phi)$ for each f in $C^b(X)$ is compact if and only if for each $\varepsilon > 0$, the image under ϕ of $\{y: |r(y)| \geq \varepsilon\}$ is finite.*

Proof. We need only prove part (b) and proceed as in the proof of Theorem 1. Assume first that there is an infinite sequence of distinct points (a_i) in the image of $\{y: |r(y)| \geq \varepsilon\}$ under the map ϕ. We choose neighborhoods N_i as in the proof of Theorem 1 without regard to the set F (i.e., without N_i disjoint from F). Then the function f_j is chosen to be one at a_{i_j} and zero on the complement of N_i with $0 \leq f_j \leq 1$. For y in Y with $r(y) \geq \varepsilon$ and $\phi(y) = a_{i_j}$ we have

$$\|Tf_j - Tf_k\| \geq \|r(f_j \circ \phi) - r(f_k \circ \phi)\| \geq |r(y)f_j(a_{i_j})| = |r(y)| \geq \varepsilon.$$

The proof of the converse is the same as the proof in Theorem 1.

We recall that (e.g., see [1, p. 313]) that an operator from E to F is called *M-weakly compact* if for every disjoint sequence (f_n) in E bounded in norm, the sequence $(\|T(f_n)\|)$ converges to zero (f is disjoint from g if $|f| \wedge |g| = 0$). We can now prove the following:

Theorem 3. *A weighted composition operator T satisfying the hypothesis of Theorem 1 is compact if and only if T is M-weakly compact.*

Proof. We will verify that T is M-weakly compact if and only if condition (ii) of Theorem 1 is satisfied. Given that (ii) is not satisfied, the same argument as in the proof of Theorem 1 yields a bounded disjoint sequence whose image under T does not converge to zero in norm. Given that (ii) is satisfied, let H be the function in the proof of Theorem 1. We note that for (f_i) a disjoint sequence bounded in E, the sequence (HTf_i) is also disjoint. Since $\phi\{y: r(y) \geq \varepsilon/2\}$ is finite, at most finitely many of the functions HTf_i can be nonzero. Thus (HTf_i) is convergent to zero in norm. Now HT is M-weakly compact and since the M-weakly compact operators are closed (e.g., [1, p. 317]), we conclude that T is M-weakly compact.

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