NOTES ON THE INVERSION OF INTEGRALS II

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(Communicated by Jonathan M. Rosenberg)

Abstract. If $W$ is a Picard bundle on the Jacobian $J$ of a curve $C$, we have the problem of describing $W$ globally. The theta divisor $\theta$ is ample on $J$. Thus it is possible to write $W$ as the sheaf associated to a graded $M$ over the well-known ring $\oplus_{m \geq 0} \Gamma(J, \mathcal{O}_J(m \theta))$. In this paper we compute the degree of generators and relations for such a module $M$.

In this part I will solve a problem which will allow the development of the normal presentation of twists of the Picard bundle on the Jacobian rather than their pull-back by isogenies. Also I will discuss the rigidity of Picard bundles pulled back by isogenies.

1. The restriction theorem

Let $\mathcal{L}$ be an ample invertible sheaf on an abelian variety $X$ over $k = \overline{k}$. Let $1 \to \mathbb{G}_m \to H \to K \to 0$ be Mumford’s theta group of $\mathcal{L}$. Here $K$ is the closed subgroupscheme of $X$ given by $\text{Ker}(\psi) = K$ where $\psi : X \to \hat{X}$ is as usual. Take a maximal closed subgroupscheme $K^1$ of $K$ such that $\alpha^{-1}(K_1)$ is abelian. As the commutative extension of $K_1$ by $\mathbb{G}_m$ splits, we may choose a homomorphism $\sigma : K_1 \to H$ such that $\alpha \circ \sigma = 1_{K_1}$.

Let $x$ be a point of $X$. Then we have a restriction homomorphism $R(x) : \Gamma(X, \mathcal{L}) \to \Gamma(x + K_1, \mathcal{L}|_{x+K_1})$. Our first result is

Lemma 1. For $x$ in a non-empty open subset of $X$, the map $R(x)$ is an isomorphism.

Proof. By standard theory both spaces have the same dimension. Better yet they are both the regular representation of $K_1$. Let’s see how the representations occur. By definition we have a given $H$-linearization of $\mathcal{L}$ and, hence, an induced $\alpha^{-1}(K_1)$-linearization of $\mathcal{L}|_{x+K_1}$. The restriction $R(x)$ is obviously a $\alpha^{-1}(K_1)$-homomorphism. Thus via $\alpha$ the restriction $R(x)$ is a homomorphism of $K_1$-modules. by [7 or 8] $\Gamma(X, \mathcal{L})$ is isomorphic to the regular representation of $K_1$. The same holds for $\Gamma(x + K_1, \mathcal{L}|_{x+K_1})$ as $\mathcal{L}|_{x+K_1}$ is a $K_1$-linearized

Received by the editors February 2, 1989, and in revised form April 10, 1989.
1980 Mathematics Subject Classification (1985 Revision). Primary 14H40; Secondary 14K30.
Key words and phrases. Algebraic curves, Jacobians and Picard bundles.

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0002-9939/90 $1.00 + .25$ per page

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invertible sheaf on a principal homogeneous space of $K_1$ (in other words $\Gamma(x + K_1, \mathcal{L}|_{x+K_1})$ is induced from the trivial one-dimensional representation of the identity subgroup of $K_1$).

To check that $R(x)$ is an isomorphism we will use the criterion

(1) $R(x)(t_\chi) \neq 0$ for all eigenvectors $t_\chi$ with eigenvalue a character $\chi$ of $K_1$.

This criterion is clear because any non-zero $K_1$-submodule $\text{Ker}(R(x))$ of $\Gamma(X, \mathcal{L})$ contains an eigenvector because $K_1$ is abelian. The next point is that up to constant multiple $t_\chi$ is determined by $\chi$ because $\Gamma(X, \mathcal{L})$ is the regular representation. Therefore we need only check a finite number of conditions for our criterion. As $t_\chi \neq 0$ it has non-zero value at most points $x$ of $X$. Therefore $R(x)(t_\chi) \neq 0$ for most $x$ and all of the finitely many $\chi$. □

Now let $\mathcal{L}$ be another ample invertible sheaf on $X$. Then we have a restriction

$$S(x) : \Gamma(X, \mathcal{L} \otimes \mathcal{M}) \to \Gamma(X + K_1, \mathcal{L} \otimes \mathcal{M}|_{x+K_1}),$$

which satisfies

Theorem 2. For all points $x$ of $X$ the restriction $S(x)$ is surjective.

Remark. When $\mathcal{M} = \mathcal{L}$ the result implies that $\Gamma(X, \mathcal{L} \otimes \mathcal{L})$ generates $\mathcal{L} \otimes \mathcal{L}$ which is well-known.

Remark. In some classical cases Theorem 2 is due to S. Koizumi and called by him the “rank theorem” [5,6].

Proof. Let $\theta$ be the zeroes of a section $\sigma$ of $\mathcal{M}$. Then for fixed $x$

$$(\theta + y) \cap (x + K_1) = \emptyset \text{ if and only if } y \in -\theta + x + K.$$  

Thus for general $y$, $T_y^*\mathcal{M}|_{x+K_1}$ is nowhere vanishing section of $T_y^*\mathcal{M}|_{x+K_1}$, where $T_y : X \to X$ is translation by $y$. Now $T_y^*\mathcal{M}$ runs through all sheaves algebraically equivalent to $\mathcal{M}$ as $\mathcal{M}$ is ample.

Thus we may find $\mathcal{M}'$ and $\mathcal{L}'$ algebraically equivalent to $\mathcal{M}$ and $\mathcal{L}$ such that $\mathcal{M}' \otimes \mathcal{L}' \cong \mathcal{M} \otimes \mathcal{L}$ so that the restriction $S(x) : \Gamma(X, \mathcal{M} \otimes \mathcal{L}) \to \Gamma(x + K_1, \mathcal{M} \otimes \mathcal{L}|_{x+K_1})$ is surjective if the restriction $R'(x) : \Gamma(X, \mathcal{L}') \to \Gamma(x + K_1, \mathcal{L}'|_{x+K_1})$ is surjective where $\mathcal{L}'$ is general of its type. This follows by multiplying a section of $\Gamma(X, \mathcal{M}')$ which vanishes nowhere on $x + K_1$.

I claim the previous lemma means that $R'(x)$ is an isomorphism for general $\mathcal{L}'$. This claim implies the theorem from the above. For the claim take $\mathcal{L}' = T_y^*\mathcal{L}$. Then $R'(x) \approx R(x - y)$. Hence the claim follows from the lemma. □

2. Global spanning

Let $\mathcal{V}(\mathcal{L}) = \pi_X^*(\pi_X^*(\mathcal{L}) \otimes \mathcal{P})$ where $\mathcal{P}$ is a Poincaré sheaf on $X \times \hat{X}$ where $\hat{X}$ is the dual abelian variety. Let $\mathcal{N}$ be an invertible sheaf on $\hat{X}$. We want to know when $\mathcal{V}(\mathcal{L}) \otimes \mathcal{N}$ is spanned by its global sections.
Let $Y$ be $X/K_1$. As $K_1 \subset \text{Ker}(\psi_\varphi)$ we have a factorization $X \xrightarrow{\alpha} Y \xrightarrow{b} \bar{X}$ of $\psi_\varphi$. Let $L$ be the closed subgroupscheme $K/K_1$ of $Y$. Let $\mathcal{E}$ be the invertible sheaf on $\bar{Y}$ gotten by descending $\mathcal{L}$ with the $K_1$-action given by the homomorphism $\sigma$. We will assume that $\text{char}(k) \nmid \deg(\psi_\varphi)$.

**Lemma 3.** $\mathcal{V}(\mathcal{L}) \otimes \mathcal{N}$ is spanned by its global sections if and only if the restriction

$$U(y) : \Gamma(Y, \mathcal{E} \otimes b^* \mathcal{N}) \to \Gamma(y + L, \mathcal{E} \otimes b^* \mathcal{N}|_{y+L})$$

is surjective for all points $y$ of $Y$.

**Proof.** For $y$ in $Y$, $\mathcal{V}(\mathcal{L}) \otimes \mathcal{N}$ is spanned by its sections at $b(y)$ if and only if the restriction $\Gamma(X, \mathcal{V}(\mathcal{L}) \otimes \mathcal{N}) \to \mathcal{V}(\mathcal{L}) \otimes \mathcal{N}(b(y))$ is surjective if and only if the pull-back plus restriction $\mathcal{W}(y) : \Gamma(X, \mathcal{V}(\mathcal{L}) \otimes \mathcal{N}) \to (b^*(\mathcal{V}(\mathcal{L}) \otimes \mathcal{N}))(y)$ is surjective. As $b$ is surjective, $\mathcal{V}(\mathcal{L}) \otimes \mathcal{N}$ is spanned by its global sections if and only if $\mathcal{W}(y)$ is surjective for all $y$ in $Y$. Thus it will be enough to prove that $\mathcal{W}(y)$ is surjective if and only if $U(y)$ is surjective. To show this I intend to give a commutative diagram

$$U(y) : \Gamma(Y, \mathcal{E} \otimes b^* \mathcal{N}) \to \Gamma(y + L, \mathcal{E} \otimes b^* \mathcal{N}|_{y+L})$$

$$\downarrow \text{U}_A$$

$$\mathcal{W}(y) : \Gamma(X, \mathcal{V}(\mathcal{L}) \otimes \mathcal{N}) \to \mathcal{V}(\mathcal{L}) \otimes \mathcal{N}(y).$$

We need to compute the sheaf $b^*(\mathcal{V}(\mathcal{L}) \otimes \mathcal{N})$ together with its $L$-linearization. Let $\chi$ be a character of $K_1$. We have a sheaf $\mathcal{E}_\chi$ on $Y$ gotten by descending the action of $K_1$ on $\mathcal{L}$ via $\sigma$ twisted by $\chi$. Thus $\mathcal{E}_1 = \mathcal{E}$ and all of the $\mathcal{E}_\chi$'s are algebraically equivalent. The first step is

**Sublemma 4.** We have a natural isomorphism

$$b^*(\mathcal{V}(\mathcal{L}) \otimes \mathcal{N}) \simeq \bigoplus_{\chi \in K_1} \Gamma(Y, \mathcal{E}_\chi) \otimes_k (\mathcal{E}_\chi \otimes b^* \mathcal{N}),$$

where the spaces $\Gamma(Y, \mathcal{E}_\chi)$ are lines.

**Proof.** As the $K_1$-module $\Gamma(X, \mathcal{L})$ is the regular representation, it is the direct sum of its $\chi$-eigenspaces $\Gamma(\chi, \mathcal{L})^\chi$ which are lines. On the other hand $\Gamma(X, \mathcal{L})^\chi = \Gamma(X/K_1, \mathcal{L}_X^\chi) = \Gamma(Y, \mathcal{E}_\chi^\chi)$. Thus the spaces are lines.

Now by [1] we have a canonical isomorphism $(\psi_\varphi)^*\mathcal{V}(\mathcal{L}) \simeq \Gamma(X, \mathcal{L}) \otimes_k \mathcal{L}_X^{\otimes -1}$ where the $K_1$-action (even $K$-action) is the obvious one. As $\psi_\varphi = b \circ a$, $b^*\mathcal{V}(\mathcal{L})$ is the sheaf of $K_1$-invariants in $\Gamma(X, \mathcal{L}) \otimes_k \mathcal{L}_X^{\otimes -1}$ which is $\bigoplus \Gamma(X, \mathcal{L})^\chi \otimes_k (\mathcal{L}_X^{\otimes -1})^\chi = \bigoplus \Gamma(Y, \mathcal{E}_\chi) \otimes_k \mathcal{E}_\chi^{\otimes -1}$. The sublemma results by tensoring this isomorphism with $b^*\mathcal{N}$. □

The second step gives rise to $L$-action under this isomorphism. For simplicity of exposition we will assume that there is another maximal subgroup $K_2$ of $K$ with a section $\tau$ of $\alpha$ over $K_2$ such that $K_2 \cap K_1 = \{0\}$. By projection $\tilde{K}_2 \approx L$ and via the Weil form $e_\varphi$ of $\mathcal{L}$, $K_2$ (and hence $L$) may be identified with
Let \( \psi(e) \) denote the character of \( K_1 \) corresponding to an element \( \ell \) of \( L \). The crucial fact is that

\[
(*) \quad \text{for any } \chi \text{ in } K_1 \text{ and } \ell \text{ in } L \text{ we have a } T_\ell \text{-isomorphism}
\]

\[
\rho(\ell, \chi): \mathcal{E}_\chi \to \mathcal{E}_{\chi \psi(\ell)} \quad \text{such that } \rho(\ell, \chi \psi(\ell_1)) \circ \rho(\ell, \chi) = \rho(\ell + \ell_2, \chi).
\]

Here the \( T_\kappa \)-isomorphism \( a^*(\rho, \chi): \mathcal{L} \to \mathcal{L} \) is just the action of the element \( \kappa \) of \( K \) over \( \ell \) on \( \mathcal{L} \) via \( \tau \). The above fact results from the study of how the \( K_2 \)-action on \( \mathcal{L} \) fails to commute with that of \( K_1 \). Thus we have isomorphism

\[
\Gamma(y, \rho(\ell, \chi)): \Gamma(Y, \mathcal{E}_\chi) \cong \Gamma(Y, \mathcal{E}_{\chi \psi(\ell)})
\]

and the \( T_\ell \)-isomorphism

\[
(\rho(-\ell, \chi \psi(\ell)) \otimes K_\ell): \mathcal{E}_\chi^{\otimes -1} \otimes b^* \mathcal{N} \to \mathcal{E}_{\chi \psi(\ell)}^{\otimes -1} \otimes b^* \mathcal{N},
\]

where \( K_\ell \) is the action of \( \ell \) on \( b^* \mathcal{N} \). Summing up without any more details we get

**Sublemma 5.** The \( L \)-action on \( \bigoplus \Gamma(y, \mathcal{E}_\chi) \otimes_k (\mathcal{E}_\chi^{\otimes -1} \otimes b^* \mathcal{N}) \) is the direct sum of the tensor products of the above isomorphisms.

Now \( \Gamma(\tilde{X}, \mathcal{V}(\mathcal{L}) \otimes \mathcal{N}) \) is the space of \( L \)-invariants in \( \Gamma(Y, b^*(\mathcal{V}(\mathcal{L}) \otimes \mathcal{N})) \). Thus

\[
\Gamma(\tilde{X}, \mathcal{V}(\mathcal{L}) \otimes \mathcal{N}) = (\bigoplus \Gamma(Y, \mathcal{E}_\chi) \otimes_k \Gamma(Y, \mathcal{E}_\chi^{\otimes -1} \otimes b^* \mathcal{N}))^L.
\]

Explicitly all such \( L \)-invariants are

\[
M(a) = \sum_{\ell} \Gamma(Y, \rho(\ell, 1)) \cdot d \times \Gamma(\rho(-\ell, \psi(\ell))^{\otimes -1} \otimes K_\ell) a,
\]

where \( \ell \) is a fixed non-zero element of \( \Gamma(y, \mathcal{E}) \) and \( a \) is an arbitrary element of \( \Gamma(Y, \mathcal{E}_\chi^{\otimes -1} \otimes b^* \mathcal{N}) \). The isomorphism \( A \) is just the correspondence between invariants and \( a \)’s.

Next we need to evaluate the section \( M(a) \) at a point \( y \) of \( Y \). The value of \( M(a) \) at \( y \) is an element of \( \bigoplus_{\ell} \Gamma(y, \mathcal{E}_\chi) \otimes_k (\mathcal{E}_\chi^{\otimes -1} \otimes b^* \mathcal{N})(y) \cong \bigoplus_{\ell} \Gamma(y, \mathcal{E}) \otimes_k (\mathcal{E}_\chi^{\otimes -1} \otimes b^* \mathcal{N})(y + \ell) = \Gamma(y, \mathcal{E}) \otimes \Gamma(y + \ell, \mathcal{E}^{\otimes -1} \otimes b^* \mathcal{N}|_{y + \ell}) \) under the isomorphism \( M(a)(y) \) goes to \( 1 \otimes \sum_{\ell} a(y + \ell) \delta_{y+\ell} \). Using these isomorphisms we have the isomorphism \( B \) and the required commutative diagram. □

Now we are ready to put together the previous results. Let \( \mathcal{R} \) be an ample invertible sheaf on \( X \) such that \( \psi_\mathcal{R}: X \to \tilde{X} \) is an isomorphism which we will take to be an identification. Thus \( X \) is principally polarized. Assume that \( \psi_\mathcal{R} = \ell_1 \mathcal{I}_X \) and \( \psi_\mathcal{N} = \mathcal{N} \mathcal{I}_X \). Then we have \( \ell \) and \( n > 0 \) as \( \mathcal{N} \) and \( \mathcal{L} \) are ample.
Theorem 6. If \( l(n - 1) > 1 \) then \( \mathcal{V}(L) \otimes \mathcal{N} \) is generated by its sections whenever \( \text{char}(k) \nmid l \).

Proof. Choose a decomposition \( K_1 \oplus K_2 \) of \( X_\ell = \text{Ker}(\ell_1 \chi) \) by subgroups which are isotropic with respect to the Weil form of \( L \). Let \( Y = X/K_1 \). Then \( Y \) is principally polarized by \( \mathcal{O} \) where \( a * \mathcal{O} \cong L \) and \( X \rightarrow Y \rightarrow X \) is the factorization of \( \ell_1 \chi \). The classifying homomorphism \( \psi_{b,*}: Y \rightarrow Y \) is \( n/l \gamma_1 \) by an elementary calculation. Thus \( \mathcal{O} \otimes b^\ast \mathcal{N} \) is algebraically equivalent to \( \mathcal{O} \otimes b^\ast \mathcal{N} \) by Lemma 3 we need to check whether \( \Gamma(Y, \mathcal{O} \otimes b^\ast \mathcal{N}) \rightarrow \Gamma(y + L, \mathcal{O} \otimes b^\ast \mathcal{N}|_{y + L}) \) is surjective for all \( y \) in \( Y \) where \( L \) is the isomorphic image of \( K_2 \) in \( Y \).

Now \( L \) is a maximal isotopic subgroup of \( Y_\ell = \text{Ker}(\psi_\mathcal{E} \otimes \ell) \) with respect to \( e_\mathcal{E} \otimes \ell \). If \( (n - 1)\ell > 1 \), \( \mathcal{O} \otimes b^\ast \mathcal{N} \otimes \mathcal{O} \otimes b^\ast \mathcal{N} \equiv \mathcal{M} \) is ample. Thus the restriction is surjective by Theorem 2. \( \square \)

3. Normal presentation for Picard bundles

We will be using the notation of Part I [4].

Theorem 7. (a) \( \mathcal{U}_n(D) \otimes \mathcal{M} \) is normally presented for \( \mathcal{R} \) if \( m \geq 3 \) and \( r \geq 4 \). If furthermore \( \text{char}(k) \nmid m \) then

(b) it is strongly normally presented and

(c) the multiplication \( \Gamma(J, \mathcal{U}_n(D) \otimes \mathcal{M}) \otimes \Gamma(J, \mathcal{R}) \rightarrow \Gamma(J, \mathcal{U}_n(D) \otimes M \otimes \mathcal{R}) \) is surjective.

Proof. The first point is that \( \mathcal{U}_n(D) \) only depends on \( \mathcal{L}_n|_C(-D) \). So choosing \( \mathcal{L}_n \) and \( D \) correctly we may assume that \( \text{char}(k) \nmid n \) and \( D \) is reduced. So by Theorem 6 \( \mathcal{U}_n \otimes \mathcal{M} \) is generated by its sections. Then we proceed as in the proof of Part I, Theorem 7. As \( \mathcal{U}_n \rightarrow \mathcal{W}_ng \) is surjective, \( \mathcal{W}_ng \otimes \mathcal{M} \) is generated by its sections. From the exact sequence

\[
0 \rightarrow \mathcal{W}_ng \otimes \mathcal{M} \rightarrow \bigoplus_{1 \leq i \leq d} \mathcal{L}_i \otimes \mathcal{M} \rightarrow \mathcal{U}_n(D) \otimes \mathcal{M} \rightarrow 0
\]

and Part I, Lemma 2 we need only see for (a) \( \mathcal{L}_i \otimes \mathcal{M} \) is strongly normally presented and for (b) \( H^1(J, \mathcal{W}_ng \otimes \mathcal{M}) = 0 \). The first statement is Part I, Theorem 6 and the second follows from [1, Theorem 3.8] when \( n > 2 \), which we may assume. This proves (a) and (b).

For (c) by the above it suffices to show that the multiplier is surjective

\[
\Gamma(J, \mathcal{L}_i \otimes \mathcal{M}) \otimes \Gamma(J, \mathcal{R}) \rightarrow \Gamma(J, \mathcal{L}_i \otimes \mathcal{M} \otimes \mathcal{R})
\]

but this follows from the Mumford-Koizumi Theorem [3]. \( \square \)

Next we compute the dimension of sections of twists of \( \mathcal{U}_n(D) \).

Theorem 8. (a) If \( m > 0 \) then \( H^i(J, \mathcal{U}_n(D) \otimes \mathcal{M}) = 0 \) if \( i > 0 \),

(b) \( \Gamma(J, \mathcal{U}_n(D) \otimes \mathcal{M}) = H^1(C, \mathcal{L}_n|_C(-D) \otimes \pi_{C*}(\pi_{C*}^\ast \mathcal{M} \otimes \mathcal{R}|_{C \times J})) \), and

(c) \( \dim \Gamma(J, \mathcal{U}_n(D) \otimes \mathcal{M}) = (d - gn + g - 1)m^g + gm^{g-1} \).
Proof. By (a) the dimension equals the Euler characteristic of $\mathcal{Z}_n(D) \otimes \mathcal{M}$, which by the Riemann-Roch Theorem is the number of points in $\text{ch}((\mathcal{Z}_n(D) \otimes \mathcal{M})$ which equals $(\text{rank} + \theta) \exp(m\theta)$. (See Theorem 8 in Part I). Thus (c) follows.

For (a) and (b) as $\mathcal{Z}_n(D) = R_{\pi_1 \ast}(\pi_1^* \mathcal{L}_n \otimes \mathcal{P}|_{C \times J}(-D \times J))$ and the other direct images are zero, we have an isomorphism $H^i(J, \mathcal{Z}_n(D) \otimes \mathcal{M}) \approx H^{i+1}(C \times J, \pi_1^* \mathcal{L}_n \otimes \mathcal{P} \otimes \pi_2^* \mathcal{M}|_{C \times J}(-D \times J))$ but as $m > 0$ the higher direct images of the last sheaf via $\pi_C$ are zero [1]. Therefore

$$H^{i+1}(C \times J, \pi_1^* \mathcal{L}_n \otimes \mathcal{P} \otimes \pi_2^* \mathcal{M}|_{C \times J}(-D \times J)) = H^{i+1}(C, \mathcal{L}_n|_C(-D) \otimes \pi_1^* (\mathcal{P} \otimes \pi_2^* \mathcal{M})|_C).$$

As $C$ is a curve the last cohomology group is zero if $i > 0$. Thus (a) and (b) follow from the two isomorphisms. □

4. RIGIDITY OF THE PICARD UNDER PULL-BACKS

As is well-known the Picard bundles $\mathcal{Z}_n(D)$ describe the fibering $f: C^{(r)} \to J$ of the symmetric product $C^{(r)}$ over the Jacobian $J$ for $r > 2g - 2$. The rigidity of $\mathcal{Z}_n(D)$ translates into a statement about the deformations of $C^{(r)}$ [2]. In the current situation we want to study the deformations of a variety $X$ where $X = C^{(r)} \times_J A$ for an isogeny $f: A \to J$ of degree prime to the characteristic. Thus $X$ is an abelian unramified covering of $C^{(r)}$ of degree prime to $\text{char}(k)$.

Theorem 9. If $r > 3$ and $C$ has general moduli then any deformation of $X$ is induced by a deformation of $C$ and a deformation of the isogeny $f$.

We will prove this theorem later. Let $\text{Pic}^0(Y)$ be the connected component of the Picard scheme of a variety $Y$. We will begin by proving

Theorem 10. The natural mapping $\text{Pic}^0(C^{(r)}) \to \text{Pic}^0(X)$ is an isogeny of abelian varieties if $r > 1$.

Proof. First recall that $\text{Pic}^0(C^{(r)})$ is isomorphic to the Jacobian $J$. A key fact is that for any two morphisms $g$ and $h: S \to J$, $(g + f)^* : \text{Pic}^0(J) \to \text{Pic}^0(S)$ is the product $g^* \otimes f^*$ (this is the theorem of the square). To use this fact look at $\text{Pic}^0(J) \to \text{Pic}(C^{(r)})^i \to \text{Pic}^0(C^{(r)}).$ Then the composition sends $[\mathcal{L}]$ to $[\otimes_i \pi_i^* (\mathcal{L}^r)|_C]$. By autoduality of the Jacobian $\text{Pic}^0(J) \approx \text{Pic}^0(C) \approx J$. Thus the composition is $J \to \text{Pic}^0(C^{(r)}) \to J$. By the fixed point argument [2, Lemma 1.3] with $G = G_m$ pull-back gives an inclusion $H^1(C^{(r)}, \mathcal{O}_{C^{(r)}}) \hookrightarrow H^1(C^{(r)}, \mathcal{O}_{C^{(r)}}^{\text{Sym}(r)}).$ Thus $i$ is an inclusion and hence an isomorphism with $J = (C^{(r)})^{\text{Sym}(r)}$ (set-theoretically). To remove this qualification we use the argument when $G = G_a$. Hence $H^1(C^{(r)}, \mathcal{O}_{C^{(r)}}) \hookrightarrow (H^1(C^{(r)}, \mathcal{O}_{C^{(r)}}^{\text{Sym}(r)})$ is injective and, hence, $\text{Pic}^0(C^{(r)}) \approx J$ because we have an isomorphism on tangent spaces.
We next apply the fixed point argument to the covering \( Y = C^{x r} \times_{C^n} X \rightarrow X \) of \( X \) with Galois group \( \text{Sym}(r) \). Then we get an injection \( \text{Pic}^0(Y) \hookrightarrow \text{Pic}^0(X)^{\text{Sym}(r)} \). We need to see that the composition \( H^1(C^{x r}, \mathcal{O}_{C^{x r}}^\chi) \rightarrow H^1(Y, \mathcal{O}_Y)^{\text{Sym}(r)} \) is an isomorphism, or, rather, \( H^1(C^{x r}, \mathcal{O}_{C^{x r}}^{\chi}) \rightarrow H^1(Y, \mathcal{O}_Y)^{\text{Sym}(r)} \) is an isomorphism. Better yet we have

Claim. \( H^1(C^{x r}, \mathcal{O}_{C^{x r}}) \rightarrow H^1(Y, \mathcal{O}_Y) \) is an isomorphism.

This uses Kummer theory. Let \( K \) be the kernel of \( f: A \rightarrow J \). Then for each character \( \chi \) of \( K \) we have an invertible sheaf \( \mathcal{O}_{C}^\chi \) on \( J \) gotten by descending the translation action of \( K \) on \( \mathcal{O}_A \) by \( \chi \). Then \( \mathcal{O}_A^\chi \) is contained in \( \text{Pic}^0(J) \). For any morphism \( k: S \rightarrow J \) we define \( \mathcal{O}_S^\chi \) to be \( k^* \mathcal{O}_A^\chi \). Then by the above key fact \( \mathcal{O}_{C^{x r}}^\chi \) is contained in \( \text{Pic}^0(X) \). Hence by the Künneth formula, if \( \chi \neq 1 \), \( H^1(C^{x r}, \mathcal{O}_{C^{x r}}^\chi) = 0 \) as \( r > 1 \) and \( H^0(C, \mathcal{O}_C^\chi) = 0 \) because \( \mathcal{O}_C^\chi \) has degree zero and is not trivial. Now \( \alpha^* \mathcal{O}_Y = \bigoplus \mathcal{O}_{C^{x r}}^\chi \) where \( \alpha: Y \rightarrow C^{x r} \) is the projection. Thus \( H^1(Y, \mathcal{O}_Y) = \bigoplus \mathcal{O}_{C^{x r}}^\chi \) is an isomorphism. This proves the claim. 

I checked that the mapping of Theorem 10 is just \( f^+ \): \( J^+ \rightarrow A^+ \).

What we will need is the same fact for \( \alpha': X \rightarrow C^{(r)} \).

Corollary 11. If \( \chi \neq 1 \), then \( H^1(C^{(r)}, \mathcal{O}_{C^{(r)}}^\chi) = 0 \) if \( r > 1 \) and \( H^0(C^{(r)}, \mathcal{O}_{C^{(r)}}) = 0 \) if \( r \geq 1 \).

Proof. The second fact is a consequence of \( \mathcal{O}_{C^{(r)}}^\chi \) being non-trivial but numerically trivial. 

Now we can start the

Proof of Theorem 9. We need to compute \( H^1(X, \theta_X^r) \). Now \( \sigma^r \theta_X^r \approx \theta_{C^{(r)}} \otimes \mathcal{O}_X^\chi = \theta_{C^{(r)}} \otimes \bigoplus \mathcal{O}_{C^{(r)}}^\chi \). Thus we have a decomposition \( H^1(X, \theta_X^r) = \bigoplus H^1(C^{(r)}, \theta_{C^{(r)}} \otimes \mathcal{O}_{C^{(r)}}^\chi) \). We need to see that

\[
\text{(A) if } \chi \neq 1 \text{ then } H^1(C^{(r)}, \theta_{C^{(r)}} \otimes \mathcal{O}_{C^{(r)}}^\chi) = 0
\]

because the theorem will follow from \( A \) by deformation theory.

We will use the method of [2]. Let \( D_r \) be the universal divisor on \( C \times C^{(r)} \). Then \( \theta_{C^{(r)}} = \tau_* (\mathcal{O}_{C^{(r)}} (D_r) \mid D_r) \) where \( \tau_r: D_r \rightarrow C^{(r)} \) is the projection. Thus \( H^1(C^{(r)}, \theta_{C^{(r)}} \otimes \mathcal{O}_{C^{(r)}}^\chi) \approx H^1(D_r, \mathcal{O}_{C^{(r)}} (D_r) \otimes \mathcal{O}_{C^{(r)}}^\chi) \). The first point is

Claim. If \( r > 2 \) and \( n \geq 0 \) then

\[
H^1(D_r, \pi_* ((\theta_C \otimes \mathcal{O}_C^\chi)^{\otimes n})(D_r) \mid D_r) \otimes \mathcal{O}_{C^{(r)}}^\chi
\]

\[
\cong H^1(D_{r-1}, \pi_* ((\theta_C \otimes \mathcal{O}_C^\chi)^{\otimes n+1})(D_{r-1}) \mid D_{r-1}) \otimes \mathcal{O}_{C^{(r-1)}}^\chi
\]
Proof of claim. We have an isomorphism
\[ \alpha_r : C \times C^{(r-1)} \xrightarrow{\cong} D_r \]
where
\[ \alpha_r^* \mathcal{O}_{C \times C^{(r)}}(D_r)|_{D_r} \simeq \pi_C^* \theta_C(D_{r-1}) \]
and
\[ \alpha_r^* \tau_r^* \mathcal{O}_{C^{(r)}}(\chi) = \pi_C^* \mathcal{O}_C(\chi) \otimes \pi_{C^{(r-1)}}^* \mathcal{O}_{C^{(r-1)}}(\chi). \]
Thus
\[ \pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi))^\otimes n)(D_r)|_{D_r} \otimes \tau_r^* \mathcal{O}_{C^{(r)}}(\chi) \]
corresponds via \( \alpha_r \) to the sheaf
\[ \left( \pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi))^\otimes n+1) \otimes \pi_{C^{(r-1)}}^* \mathcal{O}_{C^{(r-1)}}(\chi) \right)(D_{r-1}). \]
Using the sequence
\[ 0 \to \mathcal{O}_{C \times C^{(r-1)}} \to \mathcal{O}_{C \times C^{(r-1)}}(D_{r-1}) \to \mathcal{O}_{C \times C^{(r-1)}}(D_{r-1})|_{D_{r-1}} \to 0 \]
tensored by
\[ \left( \pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi))^\otimes n+1) \otimes \pi_{C^{(r-1)}}^* \mathcal{O}_{C^{(r-1)}}(\chi) \right), \]
we see that
\[ H^i(C \times C^{(r-1)}, \pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi))^\otimes n+1) \otimes \pi_{C^{(r-1)}}^* \mathcal{O}_{C^{(r-1)}}(\chi))(D_{r-1})) \]
\[ \cong H^i(D_{r-1}, \pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi))^\otimes n+1)(D_{r-1})|_{D_{r-1}} \otimes \tau_r^* \mathcal{O}_{C^{(r-1)}}(\chi)) \]
because
\[ H^i(C \times C^{(r-1)}, \pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi))^\otimes n+1) \otimes \pi_{C^{(r-1)}}^* \mathcal{O}_{C^{(r-1)}}(\chi) \]
is zero for \( i \leq 2 \). This vanishing follows from the Künneth formula from
\[ H^0(C, (\theta_C \otimes \mathcal{O}_C(\chi)))^\otimes n+1 = 0 \]
as the degree of the sheaf \(< 0 \) for \( g > 2 \) and
\[ H^i(C^{(r-1)}, \mathcal{O}_{C^{(r-1)}}(\chi)) = 0 \]
for \( 0 \leq i \leq 1 \) by Corollary 11. \( \square \)

Using the claim inductively we have an isomorphism
\[ H^1(C^{(r)}, \theta_{C^{(r)}} \otimes \mathcal{O}_{C^{(r)}}(\chi)) \]
\[ \cong H^1(D_2, \pi_1^*((\theta_C \otimes \mathcal{O}_C(\chi))^\otimes (r-2))(D_2)|_{D_2} \otimes \tau_2^* \mathcal{O}_{C^{(2)}}(\chi)) \]
which by the first step of the claim is isomorphic to
\[ H^1(C \times C, (\pi_1^*((\theta_C \otimes \mathcal{O}_C(\chi))^\otimes (r-1)) \otimes \pi_1^* \mathcal{O}_C(\chi))(\Delta)). \]
By duality on the surface \( C \times C \) this group is dual to the cokernel of multiplication
\[ m_r(\chi) : \Gamma(C, \Omega_C^{\otimes r} \otimes \mathcal{O}_C(\chi^{-1})) \otimes \Gamma(C, \Omega_C \otimes \mathcal{O}_C(\chi^{-1})) \to \Gamma(C, \Omega_C^{\otimes r+1} \otimes \mathcal{O}_C(\chi^{-r})). \]
Thus we need to have \( m_r(\chi) \) surjective.
We first need to have \( \Gamma(C, \Omega_c \otimes \mathcal{O}_c(\chi^{-1})) \) has no base points; i.e. \( \mathcal{O}_c(\chi^{-1}) \not\cong \mathcal{O}_c(c_1 - c_2) \) for two points \( c_1 \) and \( c_2 \) on \( C \). Otherwise \( \mathcal{O}_c = (\mathcal{O}_c(\chi))^{\deg f} = \mathcal{O}_c(\deg f \cdot c_1 - \deg f \cdot c_2) \). Therefore there is a morphism \( G: C \rightarrow \mathbb{P}^n \) of \( \deg = \deg f \) with only one point over both 0 and \( \infty \). By Riemann-Hurwitz formula \( G \) has \( 2g \) other ramification points. Normalizing one to be over 1, \( G \) depends on \( 2g - 1 \) parameters, but \( 2g - 1 < 3g - 3 = \dim(\text{Moduli}) \) as \( g > 2 \). Thus we have no base points for a general curve. By \( r > 3(\Omega_c^r \otimes \mathcal{O}_c(\chi^{-r+1})) \otimes (\Omega_c \otimes \mathcal{O}_c(\chi^{-1})) \) is not special \( (g > 1) \). Thus for general \( C \) the surjectivity of \( m_r(\chi) \) follows from the original Castelnuovo lemma. \( \square \)

Remark. X. Gomezmont has extended the results of [2] to all curves of genus \( > 2 \).

References