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NOTES ON THE INVERSION OF INTEGRALS II

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ABSTRACT. If W is a Picard bundle on the Jacobian J of a curve C, we have the problem of describing W globally. The theta divisor θ is ample on J. Thus it is possible to write W as the sheaf associated to a graded M over the well-known ring $\oplus_{m\geq 0}\Gamma(J,\mathscr{O}_J(m^4\theta))$. In this paper we compute the degree of generators and relations for such a module M.

In this part I will solve a problem which will allow the development of the normal presentation of twists of the Picard bundle on the Jacobian rather than their pull-back by isogenies. Also I will discuss the rigidity of Picard bundles pulled back by isogenies.

1. THE RESTRICTION THEOREM

Let $\mathscr{L}$ be an ample invertible sheaf on an abelian variety X over $k = \overline{k}$. Let $1 \to \mathbb{G}_m \to H \xrightarrow{\alpha} K \to 0$ be Mumford's theta group of $\mathscr{L}$. Here K is the closed subgroupscheme of X given by $\operatorname{Ker}(\psi_{\mathscr{L}}) = K$ where $\psi_{\mathscr{L}}\colon X \to \widehat{X}$ is as usual. Take a maximal closed subgroupscheme K_1 of K such that $\alpha^{-1}(K_1)$ is abelian. As the commutative extension of K_1 by $\mathbb{G}_m$ splits, we may choose a homomorphism $\sigma\colon K_1 \to H$ such that $\alpha\circ\sigma = 1_{K_1}$.

Let x be a point of X. Then we have a restriction homomorphism $R(x)$: $\Gamma(X,\mathscr{L}) \to \Gamma(x+K_1, \mathscr{L}|_{x+K_1})$. Our first result is

Lemma 1. *For x in a non-empty open subset of X, the map $R(x)$ is an isomorphism.*

Proof. By standard theory both spaces have the same dimension. Better yet they are both the regular representation of K_1. Let's see how the representations occur. By definition we have a given H-linearization of $\mathscr{L}$ and, hence, an induced $\alpha^{-1}(K_1)$-linearization of $\mathscr{L}|_{x+K_1}$. The restriction $R(x)$ is obviously a $\alpha^{-1}(K_1)$-homomorphism. Thus via α the restriction $R(x)$ is a homomorphism of K_1-modules. by [7 or 8] $\Gamma(X,\mathscr{L})$ is isomorphic to the regular representation of K_1. The same holds for $\Gamma(x+K_1, \mathscr{L}|_{x+K_1})$ as $\mathscr{L}|_{x+K_1}$ is a K_1-linearized

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invertible sheaf on a *principal* homogeneous space of K_1 (in other words $\Gamma(x+K_1, \mathscr{L}|_{x+K_1})$ is induced from the trivial one-dimensional representation of the identity subgroup of K_1).

To check that $R(x)$ is an isomorphism we will use the criterion

(1) $R(x)(t_\chi) \neq 0$ for all eigenvectors t_χ with eigenvalue a character χ of K_1.

This criterion is clear because any non-zero K_1-submodule $\mathrm{Ker}(R(x))$ of $\Gamma(X, \mathscr{L})$ contains an eigenvector because K_1 is abelian. The next point is that up to constant multiple t_χ is determined by χ because $\Gamma(X, \mathscr{L})$ is the regular representation. Therefore we need only check a finite number of conditions for our criterion. As $t_\chi \neq 0$ it has non-zero value at most points x of X. Therefore $R(x)(t_\chi) \neq 0$ for most x and all of the finitely many χ. □

Now let $\mathscr{M}$ be another ample invertible sheaf on X. Then we have a restriction

$$S(x): \Gamma(X, \mathscr{L} \otimes \mathscr{M}) \to \Gamma(X + K_1, \mathscr{L} \otimes \mathscr{M}|_{x+K_1}),$$

which satisfies

Theorem 2. *For all points x of X the restriction $S(x)$ is surjective.*

Remark. When $\mathscr{M} = \mathscr{L}$ the result implies that $\Gamma(X, \mathscr{L}^{\otimes 2})$ generates $\mathscr{L}^{\otimes 2}$ which is well-known.

Remark. In some classical cases Theorem 2 is due to S. Koizumi and called by him the "rank theorem" [5,6].

Proof. Let θ be the zeroes of a section σ of $\mathscr{M}$. Then for fixed x $(\theta + y) \cap (x + K_1) = \varnothing$ if and only if $y \in -\theta + x + K$. Thus for general y, $T_y^*\sigma|_{x+K_1}$ is nowhere vanishing section of $T_y^*\mathscr{M}|_{x+K_1}$, where $T_y: X \to X$ is translation by y. Now $T_y^*\mathscr{M}$ runs through all sheaves algebraically equivalent to $\mathscr{M}$ as $\mathscr{M}$ is ample.

Thus we may find $\mathscr{M}'$ and $\mathscr{L}'$ algebraically equivalent to $\mathscr{M}$ and $\mathscr{L}$ such that $\mathscr{M}' \otimes \mathscr{L}' \approx \mathscr{M} \otimes \mathscr{L}$ so that the restriction $S(x): \Gamma(X, \mathscr{M} \otimes \mathscr{L}) \to \Gamma(x + K_1, \mathscr{M} \otimes \mathscr{L})|_{x+K_1})$ is surjective if the restriction $R'(x): \Gamma(X, \mathscr{L}') \to \Gamma(x + K_1, \mathscr{L}'|_{x+K_1})$ is surjective where $\mathscr{L}'$ is general of its type. This follows by multiplying a section of $\Gamma(X, \mathscr{M}')$ which vanishes nowhere on $x + K_1$.

I claim the previous lemma means that $R'(x)$ is an isomorphism for general $\mathscr{L}'$. This claim implies the theorem from the above. For the claim take $\mathscr{L}' = T_y^*\mathscr{L}$. Then $R'(x) \approx R(x - y)$. Hence the claim follows from the lemma. □

2. Global spanning

Let $\mathscr{V}(\mathscr{L}) = \pi_{\widehat{X}*}(\pi_X^*(\mathscr{L}) \otimes \mathscr{P})$ where $\mathscr{P}$ is a Poincaré sheaf on $X \times \widehat{X}$ where $\widehat{X}$ is the dual abelian variety. Let $\mathscr{N}$ be an invertible sheaf on $\widehat{X}$. We want to know when $\mathscr{V}(\mathscr{L}) \otimes \mathscr{N}$ is spanned by its global sections.

Let Y be X/K_1. As $K_1 \subset \operatorname{Ker}(\psi_{\mathscr{L}})$ we have a factorization $X \xrightarrow{\alpha} Y \xrightarrow{b} \widehat{X}$ of $\psi_{\mathscr{L}}$. Let L be the closed subgroupscheme K/K_1 of Y. Let $\mathscr{Q}$ be the invertible sheaf on y gotten by descending $\mathscr{L}$ with the K_1-action given by the homomorphism σ. We will assume that $\operatorname{char}(k) \nmid \deg(\psi_{\mathscr{L}})$.

Lemma 3. *$\mathscr{V}(\mathscr{L}) \otimes \mathscr{N}$ is spanned by its global sections if and only if the restriction*

$$U(y)\colon \Gamma(Y, \mathscr{Q}^{\otimes -1} \otimes b^*\mathscr{N}) \to \Gamma(y+L, \mathscr{Q}^{\otimes -1} \otimes b^*\mathscr{N}|_{y+L})$$

is surjective for all points y of Y.

Proof. For y in Y, $\mathscr{V}(\mathscr{L}) \otimes \mathscr{N}$ is spanned by its sections at $b(y)$ if and only if the restriction $\Gamma(X, \mathscr{V}(\mathscr{L}) \otimes \mathscr{N}) \to \mathscr{V}(\mathscr{L}) \otimes \mathscr{N}(b(y))$ is surjective if and only if the pull-back plus restriction $W(y)\colon \Gamma(X, \mathscr{V}(\mathscr{L}) \otimes \mathscr{N}) \to (b^*(\mathscr{V}(\mathscr{L}) \otimes \mathscr{N})(y)$ is surjective. As b is surjective, $\mathscr{V}(\mathscr{L}) \otimes \mathscr{N}$ is spanned by its global sections if and only if $W(y)$ is surjective for all y in Y. Thus it will be enough to prove that $W(y)$ is surjective if and only if $U(y)$ is surjective. To show this I intend to give a commutative diagram

$$\begin{array}{ccc} U(y)\colon \Gamma(Y, \mathscr{Q}^{\otimes -1} \otimes b^*\mathscr{N}) & \longrightarrow & \Gamma(y+L, \mathscr{Q}^{\otimes -1} \otimes b^*\mathscr{N}|_{y+L} \\ \Big\downarrow \wr A & & \Big\downarrow \wr B \\ W(y)\colon \Gamma(X, \mathscr{V}(\mathscr{L}) \otimes \mathscr{N}) & \longrightarrow & b^*(\mathscr{V}(\mathscr{L}) \otimes \mathscr{N})(y). \end{array}$$

We need to compute the sheaf $b^*(\mathscr{V}(\mathscr{L}) \otimes \mathscr{N})$ together with its L-linearization. Let χ be a character of K_1. We have a sheaf $\mathscr{Q}_\chi$ on Y gotten by descending the action of K_1 on $\mathscr{L}$ via σ twisted by χ. Thus $\mathscr{Q}_1 = \mathscr{Q}$ and all of the $\mathscr{Q}$'s are algebraically equivalent. The first step is

Sublemma 4. *We have a natural isomorphism*

$$b^*(\mathscr{V}(\mathscr{L}) \otimes \mathscr{N}) \simeq \bigoplus_{\chi \in K_1} \Gamma(Y, \mathscr{Q}_\chi) \otimes_k (\mathscr{Q}_\chi^{\otimes -1} \otimes b^*\mathscr{N}),$$

where the spaces $\Gamma(Y, \mathscr{Q}_\chi)$ are lines.

Proof. As the K_1-module $\Gamma(X, \mathscr{L})$ is the regular representation, it is the direct sum of its χ-eigenspaces $\Gamma(\chi, \mathscr{L})^\chi$ which are lines. On the other hand $\Gamma(X, \mathscr{L})^\chi = \Gamma(X/K_1, \mathscr{L}^\chi) = \Gamma(Y, \mathscr{Q}_\chi)$. Thus the spaces are lines.

Now by [1] we have a canonical isomorphism $(\psi_{\mathscr{L}})^*\mathscr{V}(\mathscr{L}) \approx \Gamma(X, \mathscr{L}) \otimes_k \mathscr{L}^{\otimes -1}$ where the K_1-action (even K-action) is the obvious one. As $\psi_{\mathscr{L}} = b \circ a$, $b^*\mathscr{V}(\mathscr{L})$ is the sheaf of K_1-invariants in $\Gamma(X, \mathscr{L}) \otimes_k \mathscr{L}^{\otimes -1}$ which is $\oplus_\chi \Gamma(X, \mathscr{L})^\chi \otimes_k (\mathscr{L}^{\otimes -1})^{\chi^{-1}} = \oplus_\chi \Gamma(X, \mathscr{L})^\chi \otimes_k (\mathscr{L}^\chi)^{\otimes -1} = \oplus \Gamma(Y, \mathscr{Q}_\chi) \otimes_k \mathscr{Q}_\chi^{\otimes -1}$. The sublemma results by tensoring this isomorphism with $b^*\mathscr{N}$. □

The second step gives rise to L-action under this isomorphism. For simplicity of exposition we will assume that there is another maximal subgroup K_2 of K with a section τ of α over K_2 such that $K_2 \cap K_1 = \{0\}$. By projection $K_2 \approx L$ and via the Weil form $e_{\mathscr{L}}$ of $\mathscr{L}$, K_2 (and hence L) may be identified with

$\widehat{K}_1$. Let $\psi(e)$ denote the character of K_1 corresponding to an element ℓ of L. The crucial fact is that

(*) for any χ in K_1 and ℓ in L we have a T_e-isomorphism
$\rho(\ell, \chi): \mathcal{Q}_\chi \to \mathcal{Q}_{\chi\psi(\ell)}$ such that $\rho(\ell_2, \chi\psi(\ell_1)) \circ \rho(\ell_1, \chi) = \rho(\ell_1 + \ell_2, \chi)$.

Here The T_k-isomorphism $a^*(\rho, \chi): \mathcal{L} \to \mathcal{L}$ is just the action of the element k of K over ℓ on $\mathcal{L}$ via τ. The above fact results from the study of how the K_2-action on $\mathcal{L}$ falls to commute with that of K_1. Thus we have isomorphism

$$\Gamma(y, \rho(\ell, \chi)): \Gamma(Y, \mathcal{Q}_\chi) \overset{\approx}{\to} \Gamma(Y, \mathcal{Q}_{\chi\psi(\ell)})$$

and the T_e-isomorphism

$$(\rho(-\ell, \chi\psi(\ell)) \otimes K_e: \mathcal{Q}_\chi^{\otimes -1} \otimes b^*\mathcal{N} \to \mathcal{Q}_{\chi\psi(\ell)}^{\otimes -1} \otimes b^*\mathcal{N},$$

where K_ℓ is the action of ℓ on $b^*\mathcal{N}$. Summing up without any more details we get

Sublemma 5. *The L-action on $\bigoplus_\chi \Gamma(y, \mathcal{Q}_\chi) \otimes_k (\mathcal{Q}_\chi^{\otimes -1} \otimes b^*\mathcal{N})$ is the direct sum of the tensor products of the above isomorphisms.*

Now $\Gamma(\widehat{X}, \mathcal{V}(\mathcal{L}) \otimes \mathcal{N})$ is the space of L-invariants in $\Gamma(Y, b^*(\mathcal{V}(\mathcal{L}) \otimes \mathcal{N}))$. Thus

$$\Gamma(\widehat{X}, \mathcal{V}(\mathcal{L}) \otimes \mathcal{N}) = (\bigoplus_\chi \Gamma(Y, \mathcal{Q}_\chi) \otimes_k \Gamma(Y, \mathcal{Q}_\chi^{\otimes -1} \otimes b^*\mathcal{N}))^L.$$

Explicitly all such L-invariants are

$$M(a) = \sum_\ell \Gamma(Y, \rho(\ell, 1)) \cdot d \times \Gamma(\rho(-\ell, \psi(\ell))^{\otimes -1} \otimes K_e)a,$$

where ℓ is a fixed non-zero element of $\Gamma(y, \mathcal{Q})$ and a is an arbitrary element of $\Gamma(Y, \mathcal{Q}^{\otimes -1} \otimes b^*\mathcal{N})$. The isomorphism A is just the correspondence between invariants and a's.

Next we need to evaluate the section $M(a)$ at a point y of Y. The value of $M(a)$ at y is an element of $\bigoplus_\ell \Gamma(Y, \mathcal{Q}_{\psi(\ell)}) \otimes_k (\mathcal{Q}_{\psi(\ell)}^{\otimes -1} \otimes b^*\mathcal{N})(y) \approx \bigoplus_\ell \Gamma(Y, \mathcal{Q}) \otimes_k (\mathcal{Q}^{\otimes -1} \otimes b^*\mathcal{N})(y + \ell) = \Gamma(Y, \mathcal{Q}) \otimes \Gamma(y + L, \mathcal{Q}^{\otimes -1} \otimes b^*\mathcal{N}|_{y+L})$ under the isomorphism $M(a)(y)$ goes to $1 \otimes \sum_{\ell \in L} a(y + \ell)\delta_\ell$. Using these isomorphisms we have the isomorphism B and the required commutative diagram. □

Now we are ready to put together the previous results. Let $\mathcal{R}$ be an ample invertible sheaf on X such that $\psi_{\mathcal{R}}: X \to \widehat{X}$ is an isomorphism which we will take to be an identification. Thus X is principally polarized. Assume that $\psi_{\mathcal{L}} = \ell 1_X$ and $\psi_{\mathcal{N}} = \mathcal{N} 1_X$. Then we have ℓ and $n > 0$ as $\mathcal{N}$ and $\mathcal{L}$ are ample.

Theorem 6. *If* $\ell(n-1) > 1$ *then* $\mathscr{V}(\mathscr{L}) \otimes \mathscr{N}$ *is generated by its sections whenever* $\mathrm{char}(k) \nmid \ell$.

Proof. Choose a decomposition $K_1 \oplus K_2$ of $X_\ell = \mathrm{Ker}(\ell 1_X)$ by subgroups which are isotropic with respect to the Weil form of $\mathscr{L}$. Let $Y = X/K_1$. Then Y is principally polarized by $\mathscr{Q}$ where $a^*\mathscr{Q} \approx \mathscr{L}$ and $X \xrightarrow{a} Y \xrightarrow{b} X$ is the factorization of $\ell 1_X$. The classifying homomorphism $\psi_{b^*\mathscr{N}}: Y \to Y$ is $n\ell 1_Y$ by an elementary calculation. Thus $\mathscr{Q}^{\otimes -1} \otimes b^*\mathscr{N}$ is algebraically equivalent to $\mathscr{Q}^{n\ell -1}$. By Lemma 3 we need to check whether $\Gamma(Y, \mathscr{Q}^{\otimes -1} \otimes b^*\mathscr{N}) \to \Gamma(y + L, \mathscr{Q}^{\otimes -1} \otimes b^*\mathscr{N}|_{y+L})$ is surjective for all y in Y where L is the isomorphic image of K_2 in Y.

Now L is a maximal isotopic subgroup of $Y_\ell = \mathrm{Ker}(\psi_{\mathscr{Q}} \otimes \ell)$ with respect to $e_{\mathscr{Q}} \otimes \ell$. If $(n-1)\ell > 1$, $(\mathscr{Q}^{\otimes -\ell} \otimes b^*\mathscr{N} \otimes \mathscr{Q}^{\otimes -1}) \equiv \mathscr{M}$ is ample. Thus the restriction is surjective by Theorem 2. □

3. Normal presentation for Picard bundles

We will be using the notation of Part I [4].

Theorem 7. (a) $\mathscr{U}_n(D) \otimes \mathscr{M}$ *is normally presented for* $\mathscr{R}$ *if* $m \geq 3$ *and* $r \geq 4$. *If furthermore* $\mathrm{char}(k) \nmid m$ *then*

(b) *it is strongly normally presented and*

(c) *the multiplication* $\Gamma(J, \mathscr{U}_n(D) \otimes \mathscr{M}) \otimes \Gamma(J, \mathscr{R}) \to \Gamma(J, \mathscr{U}_n(D) \otimes M \otimes \mathscr{R})$ *is surjective.*

Proof. The first point is that $\mathscr{U}_n(D)$ only depends on $\mathscr{L}_n|_C(-D)$. So choosing $\mathscr{L}_n$ and D correctly we may assume that $\mathrm{char}(k) \nmid n$ and D is reduced. So by Theorem 6 $\mathscr{V}_n \otimes \mathscr{M}$ is generated by its sections. Then we proceed as in the proof of Part I, Theorem 7. As $\mathscr{V}_n \to \mathscr{W}_{ng}$ is surjective, $\mathscr{W}_{ng} \otimes \mathscr{M}$ is generated by its sections. From the exact sequence

$$0 \to \mathscr{W}_{ng} \otimes \mathscr{M} \to \bigoplus_{1 \leq i \leq d} \mathscr{S}_i \otimes \mathscr{M} \to \mathscr{U}_n(D) \otimes \mathscr{M} \to 0$$

and Part I, Lemma 2 we need only see for (a) $\mathscr{S}_i \otimes \mathscr{M}$ is strongly normally presented and for (b) $H^1(J, \mathscr{W}_{ng} \otimes \mathscr{M}) = 0$. The first statement is Part I, Theorem 6 and the second follows from [1, Theorem 3.8] when $n > 2$, which we may assume. This proves (a) and (b).

For (c) by the above it suffices to show that the multiplier is surjective

$$\Gamma(J, \mathscr{S}_i \otimes \mathscr{M}) \otimes \Gamma(J, \mathscr{R}) \to \Gamma(J, \mathscr{S}_i \otimes \mathscr{M} \otimes \mathscr{R})$$

but this follows from the Mumford-Koizumi Theorem [3]. □

Next we compute the dimension of sections of twists of $\mathscr{U}_n(D)$.

Theorem 8. (a) *If* $m > 0$ *then* $H^i(J, \mathscr{U}_n(D) \otimes \mathscr{M}) = 0$ *if* $i > 0$,

(b) $\Gamma(J, \mathscr{U}_n(D) \otimes \mathscr{M}) = H^1(C, \mathscr{L}_n|_C(-D) \otimes \pi_{C*}(\pi_2^*\mathscr{M} \otimes \mathscr{P}|_{C \times J}))$, *and*

(c) $\dim \Gamma(J, \mathscr{U}_n(D) \otimes \mathscr{M}) = (d - gn + g - 1)m^g + gm^{g-1}$.

Proof. By (a) the dimension equals the Euler characteristic of $\mathscr{U}_n(D) \otimes \mathscr{M}$, which by the Riemann-Roch Theorem is the number of points in $\mathrm{ch}(\mathscr{U}_n(D)\otimes\mathscr{M})$ which equals $(\mathrm{rank} + \theta)\exp(m\theta)$. (See Theorem 8 in Part I). Thus (c) follows.

For (a) and (b) as $\mathscr{U}_n(D) = R^1_{\pi_J *}(\pi_1^*\mathscr{L}_n \otimes \mathscr{P}|_{C\times J}(-D \times J))$ and the other direct images are zero, we have an isomorphism $H^i(J, \mathscr{U}_n(D)\otimes\mathscr{M}) \approx H^{i+1}(C \times J, \pi_1^*\mathscr{L}_n \otimes \mathscr{P} \otimes \pi_2^*\mathscr{M}|_{C\times J}(-D \times J))$ but as $m > 0$ the higher direct images of the last sheaf via π_C are zero [1]. Therefore

$$\begin{aligned} H^{i+1}&(C \times J, \pi_1^*\mathscr{L}_n \otimes \mathscr{P} \otimes \pi_2^*\mathscr{M}|_{C\times J}(-D \times J)) \\ &= H^{i+1}(C, \mathscr{L}_n|_C(-D) \otimes \pi_{1*}(\mathscr{P} \otimes \pi_2^*\mathscr{M})|_C). \end{aligned}$$

As C is a curve the last cohomology group is zero if $i > 0$. Thus (a) and (b) follow from the two isomorphisms. □

4. Rigidity of the Picard under pull-backs

As is well-known the Picard bundles $\mathscr{U}_n(D)$ describe the fibering $\int : C^{(r)} \to J$ of the symmetric product $C^{(r)}$ over the Jacobian J for $r > 2g-2$. The rigidity of $\mathscr{U}_n(D)$ translates into a statement about the deformations of $C^{(r)}$ [2]. In the current situation we want to study the deformations of a variety X where $X = C^{(r)} \times_J A$ for an isogeny $f \colon A \to J$ of degree prime to the characteristic. Thus X is an abelian unramified covering of $C^{(r)}$ of degree prime to $\mathrm{char}(k)$.

Theorem 9. *If $r > 3$ and C has general moduli then any deformation of X is induced by a deformation of C and a deformation of the isogeny f.*

We will prove this theorem later. Let $\mathrm{Pic}^0(Y)$ be the connected component of the Picard scheme of a variety Y. We will begin by proving

Theorem 10. *The natural mapping $\mathrm{Pic}^0(C^{(r)}) \to \mathrm{Pic}^0(X)$ is an isogeny of abelian varieties if $r > 1$.*

Proof. First recall that $\mathrm{Pic}^0(C^{(r)})$ is isomorphic to the Jacobian J. A key fact is that for any two morphisms g and $h \colon S \to J$, $(g+f)^* \colon \mathrm{Pic}^0(J) \to \mathrm{Pic}^0(S)$ is the product $g^* \otimes f^*$ (this is the theorem of the square). To use this fact look at $\mathrm{Pic}^0(J) \to \mathrm{Pic}(C^{(r)})^i \to \mathrm{Pic}^0(C^{\times r})$. Then the composition sends $[\mathscr{L}]$ to $[\bigotimes_i \pi_i^*(\mathscr{L}|_C)]$. By autoduality of the Jacobian $\mathrm{Pic}^0(J) \approx \mathrm{Pic}^0(C) \approx J$. Thus the composition is $J \to \mathrm{Pic}^*(C^{(r)}) \to J$. By the fixed point argument [2, Lemma 1.3] with $G = \mathbb{G}_m$ pull-back gives an inclusion $H^1(C^{(r)}, \mathscr{O}^*_{C^{(r)}}) \hookrightarrow (H^1(C^{\times r}, \mathscr{O}^*_{C^{\times r}})^{\mathrm{Sym}(r)}$. Thus i is an inclusion and hence an isomorphism with $J = (J^{\times r})^{\mathrm{Sym}(r)}$ (set-theoretically). To remove this qualification we use the argument when $G = \mathbb{G}_a$. Hence $H^1(C^{(r)}, \mathscr{O}_{C^{(r)}}) \hookrightarrow (H^1(C^{\times r}, \mathscr{O}_{C^{\times r}})^{\mathrm{Sym}(r)}$ is injective and, hence, $\mathrm{Pic}^0(C^{(r)}) \approx J$ because we have an isomorphism on tangent spaces.

We next apply the fixed point argument to the covering $Y = C^{\times r} \times_{C^{(r)}} X \to X$ of X with Galois group $\mathrm{Sym}(r)$. Then we get an injection $\mathrm{Pic}^0(X) \hookrightarrow \mathrm{Pic}^0(Y)^{\mathrm{Sym}(r)}$. We need to see that the composition $H^1(C^{(r)}, \mathcal{O}_{C^{(r)}}) \to H^1(X, \mathcal{O}_X) \hookrightarrow H^1(Y, \mathcal{O}_Y)^{\mathrm{Sym}(r)}$ is an isomorphism, or, rather, $H^1(C^{\times r}, \mathcal{O}_{C^{\times r}})^{\mathrm{Sym}(r)} \to H^1(Y, \mathcal{O}_Y)^{\mathrm{Sym}(r)}$ is an isomorphism. Better yet we have

Claim. $H^1(C^{\times r}, \mathcal{O}_{C^{\times r}}) \to H^1(Y, \mathcal{O}_Y)$ is an isomorphism.

This uses Kummer theory. Let K be the kernel of $f\colon A \to J$. Then for each character χ of K we have an invertible sheaf $\mathcal{O}_J(\chi)$ on J gotten by descending the translation action of K on $\mathcal{O}_K$ by χ. Then $\mathcal{O}_J(\chi)$ is contained in $\mathrm{Pic}^0(J)$. For any morphism $k\colon S \to J$ we define $\mathcal{O}_S(\chi)$ to be $k^*\mathcal{O}_J(\chi)$. Then by the above key fact $\mathcal{O}_{C^{\times r}}(\chi) = \bigotimes_i \pi_i^* \mathcal{O}_C(\chi)$. Hence by the Künneth formula, if $\chi \neq 1$, $H^1(C^{\times r}\mathcal{O}_{C^{\times r}}(\chi)) = 0$ as $r > 1$ and $H^0(C, \mathcal{O}_C(\chi)) = 0$ because $\mathcal{O}_C(\chi)$ has degree zero and is not trivial. Now $\alpha^*\mathcal{O}_Y = \bigoplus_\chi \mathcal{O}_{C^{\times r}}(\chi)$ where $\alpha\colon Y \to C^{\times r}$ is the projection. Thus $H^1(Y, \mathcal{O}_Y) = \bigoplus_\chi (H^1(C^{\times r}, \mathcal{O}_{C^{\times r}}(\chi)) = H^1(C^{\times r}, \mathcal{O}_{C^{\times r}})$. This proves the claim. □

I checked that the mapping of Theorem 10 is just $f^\frown\colon J^\frown \to A^\frown$.

What we will need is the same fact for $\alpha'\colon X \to C^{(r)}$.

Corollary 11. *If* $\chi \neq 1$, *then* $H^1(C^{(r)}, \mathcal{O}_{C^{(r)}}(\chi)) = 0$ *if* $r > 1$ *and* $H^0(C^{(r)}, \mathcal{O}_{C^{(r)}}(\chi)) = 0$ *if* $r \geq 1$.

Proof. The second fact is a consequence of $\mathcal{O}_{C^{(r)}}(\chi)$ being non-trivial but numerically trivial. □

Now we can start the

Proof of Theorem 9. We need to compute $H^1(X, \theta_X)$. Now $\sigma_*\theta_X \approx \theta_{C^{(r)}} \otimes \sigma_*\mathcal{O}_X = \theta_{C^{(r)}} \otimes \bigoplus_\chi \mathcal{O}_{C^{(r)}}(\chi)$. Thus we have a decomposition $H^1(X, \theta_X) = \bigoplus_\chi H^1(C^{(r)}, \theta_{C^{(r)}} \otimes \mathcal{O}_{C^{(r)}}(\chi))$. We need to see that

$$\text{(A)} \qquad \text{if } \chi \neq 1 \text{ then } H^1(C^{(r)}, \theta_{C^{(r)}} \otimes \mathcal{O}_{C^{(r)}}(\chi)) = 0$$

because the theorem will follow from A by deformation theory.

We will use the method of [2]. Let D_r be the universal divisor on $C \times C^{(r)}$. Then $\theta_{C^{(r)}} = \tau_*(\mathcal{O}_{C \times C^{(r)}}(D_r)|_{D_r})$ where $\tau_v\colon D_r \to C^{(r)}$ is the projection. Thus $H^i(C^{(r)}, \theta_{C^{(r)}} \otimes \mathcal{O}_{C^{(r)}}(\chi)) \simeq H^i(D_r, \mathcal{O}_{C \times C^{(r)}}(D_r|_{D_r} \otimes \tau_r^*\mathcal{O}_{C^{(r)}}(\chi))$. The first point is

Claim. If $r > 2$ and $n \geq 0$ then

$$H^1(D_r, \pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi)^{\otimes n})(D_r)|_{D_r} \otimes \tau_r^*\mathcal{O}_{C^{(r)}}(\chi))$$
$$\xrightarrow{\approx} H^1(D_{r-1}, \pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi)^{\otimes n+1})(D_{r-1})|_{D_{r-1}} \otimes \tau_{v-1}^*\mathcal{O}_{C^{(r-1)}}(\chi)).$$

Proof of claim. We have an isomorphism

$$\alpha_r\colon C \times C^{(r-1)} \overset{\approx}{\to} D_r$$

where

$$\alpha_r^*(\mathcal{O}_{C\times C^{(r)}}(D_r)|_{D_r}) \simeq \pi_C^*\theta_C(D_{r-1})$$

and

$$\alpha_r^*\tau_r^*\mathcal{O}_{C^{(r)}}(\chi) = \pi_C^*\mathcal{O}_C(\chi) \otimes \pi_{C^{(r-1)}}^*\mathcal{O}_{C^{(r-1)}}(\chi).$$

Thus

$$\pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi))^{\otimes n})(D_r)|_{D_r} \otimes \tau_r^*\mathcal{O}_{C^{(r)}}(\chi)$$

corresponds via α_r to the sheaf

$$(\pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi))^{\otimes n+1}) \otimes \pi_{C^{(r-1)}}^*\mathcal{O}_{C^{(r-1)}}(\chi))(D_{r-1}).$$

Using the sequence

$$0 \to \mathcal{O}_{C\times C^{(-1)}} \to \mathcal{O}_{C\times C^{(r-1)}}(D_{r-1}) \to \mathcal{O}_{C\times C^{(r-1)}}(D_{r-1})|_{D_{r-1}} \to 0$$

tensored by

$$(\pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi)^{\otimes n+1}) \otimes \pi_{C^{(r-1)}}^*\mathcal{O}_{C^{(r-1)}}(\chi),$$

we see that

$$H^1(C \times C^{(r-1)}, (\pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi)^{\otimes n+1}) \otimes \pi_{C^{(r-1)}}^*\mathcal{O}_{C^{(r-1)}}(\chi))(D_{r-1}))$$
$$\overset{\approx}{\to} H^1(D_{r-1}, \pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi))^{\otimes n+1}(D_{r-1})|_{D_{r-1}} \otimes \tau_{r-1}^*\mathcal{O}_{C^{(r-1)}}(\chi))$$

because

$$H^i(C \times C^{(r-1)}, \pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi))^{\otimes n+1} \otimes \pi_{C^{(r-1)}}^*\mathcal{O}_{C^{(r-1)}}(\chi)$$

is zero for $i \leq 2$. This vanishing follows from the Künneth formula from $H^0(C, (\theta_C \otimes \mathcal{O}_C(\chi)))^{\otimes n+1} = 0$ as the degree of the sheaf < 0 for $g > 2$ and $H^i(C^{(r-1)}, \mathcal{O}_{C^{(r-1)}}(\chi)) = 0$ for $0 \leq i \leq 1$ by Corollary 11. □

Using the claim inductively we have an isomorphism $H^1(C^{(r)}, \theta_{C^{(r)}} \otimes \mathcal{O}_{C^{(r)}}(\chi)) \overset{\approx}{\to} H^1(D_2, \pi_C^*((\theta_C \otimes \mathcal{O}_C(\chi))^{\otimes(r-2)}(D_2)|_D \otimes \tau_2^*\mathcal{O}_{C^{(2)}}(\chi))$ which by the first step of the claim is isomorphic to $H^1(C\times C, (\pi_1^*((\theta_C \otimes \mathcal{O}_C(\chi))^{\otimes r-1}) \otimes \pi_2^*\mathcal{O}_C(\chi))(\Delta))$. By duality on the surface $C \times C$ this group is dual to the cokernel of multiplication $m_r(\chi)\colon \Gamma(C, \Omega_C^{\otimes r} \otimes \mathcal{O}_C(\chi^{-r+1})) \otimes \Gamma(C, \Omega_C \otimes \mathcal{O}_C(\chi^{-1})) \to \Gamma(C, \Omega_C^{\otimes r+1} \otimes \mathcal{O}_C(\chi^{-r}))$. Thus we need to have $m_r(\chi)$ surjective.

We first need to have $\Gamma(C, \Omega_C \otimes \mathcal{O}_C(\chi^{-1}))$ has no base points; i.e. $\mathcal{O}_C(\chi^{-1}) \not\approx \mathcal{O}_C(c_1 - c_2)$ for two points c_1 and c_2 on C. Otherwise $\mathcal{O}_C = (\mathcal{O}_C(\chi))^{\deg f} = \mathcal{O}_C(\deg f \cdot c_1 - \deg f \cdot c_2)$. Therefore there is a morphism $G: C \to \mathbb{P}^n$ of $\deg = \deg f$ with only one point over both 0 and ∞. By Riemann-Hurwitz formula G has $2g$ other ramification points. Normalizing one to be over 1, G depends on $2g-1$ parameters, but $2g-1 < 3g-3 = \dim(\text{Moduli})$ as $g > 2$. Thus we have no base points for a general curve. By $r > 3(\Omega_C^{\otimes r} \otimes \mathcal{O}_C(\chi^{-r+1})) \otimes (\Omega_C \otimes \mathcal{O}_C(\chi^{-1}))$ is not special $(g > 1)$. Thus for general C the surjectivity of $m_r(\chi)$ follows from the original Castelnuovo lemma. □

Remark. X. Gomezmont has extended the results of [2] to all curves of genus > 2.

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