PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 1, January 1990

FINITE SIMPLE ABELIAN ALGEBRAS ARE STRICTLY SIMPLE

MATTHEW A. VALERIOTE

(Communicated by Maurice Auslander)

ABSTRACT. A finite universal algebra is called strictly simple if it is simple and has no nontrivial subalgebras. An algebra is said to be Abelian if for every term $t(x, \bar{y})$ and for all elements a, b, $\bar{c}$, $\bar{d}$, we have the following implication: $t(a, \bar{c}) = t(a, \bar{d}) \rightarrow t(b, \bar{c}) = t(b, \bar{d})$. It is shown that every finite simple Abelian universal algebra is strictly simple. This generalizes a well-known fact about Abelian groups and modules.

1. INTRODUCTION

It is well known and easy to prove that every simple module or Abelian group has only trivial subalgebras. In universal algebra an algebra **A** is said to be simple if it has exactly two congruences, 0_A and 1_A, the identity and universal relations, respectively. A finite simple algebra that has no proper subalgebras containing more than one element is called strictly simple. In this paper we generalize the known result for Abelian groups and modules and prove that every finite simple Abelian (universal) algebra is strictly simple.

Strictly simple algebras play a role in the investigation of minimal (equationally complete) locally finite varieties. A variety is a class of algebras that is defined by a set of equations. Equivalently, it is a class that is closed under the operations of taking direct products, subalgebras and homomorphic images of its members. It is locally finite if every one of its finitely generated members is finite. A variety is minimal if it contains no proper nontrivial subvariety.

One can easily show that any minimal locally finite variety is generated by a strictly simple algebra. In the congruence modular setting the converse is almost true. A variety is said to be congruence modular (distributive) if the congruence lattice of each of its members satisfies the modular (distributive) law for lattices. We have that if **A** is strictly simple and generates a congruence modular variety $\mathscr{V}$, then either $\mathscr{V}$ is congruence distributive and hence by Jónsson's Theorem, minimal, or $\mathscr{V}$ is Abelian and minimal, or contains exactly one nontrivial proper subvariety. The reader can consult [3] for details of this result.

Received by the editors February 14, 1989 and, in revised form, March 31, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 08A05; Secondary 03C13.
Support of NSERC is gratefully acknowledged.

We rely on [1] and [4] for most of the terminology used in this paper.

Definition 1.1. (1) An algebra $\mathbf{A}$ is called strictly simple if A is finite, $\mathbf{A}$ is simple, and $\mathbf{A}$ has no proper subalgebras containing more than one element.

(2) An algebra $\mathbf{A}$ is called Abelian if for all terms $t(x, \bar{y})$ of $\mathbf{A}$ and for all a, b, $\bar{u}$ and $\bar{v}$ in A we have

$$t(a, \bar{u}) = t(a, \bar{v}) \to t(b, \bar{u}) = t(b, \bar{v}).$$

A variety is called Abelian if all of its members are Abelian.

The above-defined Abelian property has turned out to be a very successful and useful generalization from commutative groups to general algebras. It is not hard to show that a group is Abelian in the above sense if and only if it is commutative. For example, consider the following term in the language of groups:

$$t(x, y, z) = y \cdot x \cdot z.$$

For a, b from some Abelian group $\mathbf{G}$, we have the equality $t(a^{-1}, a, 1) = t(a^{-1}, 1, a)$ and so by applying the Abelian property to this equality, we also have

$$t(b, a, 1) = t(b, 1, a),$$

i.e. $a \cdot b = b \cdot a$. Thus the group is commutative.

Using the modular commutator it has been shown that if an algebra is Abelian and generates a congruence modular variety then the algebra has polynomial operations that define an Abelian group structure on the algebra that is in some sense compatible with the operations of the algebra. See [3] for details. The Abelian property has come to take on a major role in the investigation of the structure of finite algebras and the varieties they generate, especially with the advent of the theory called tame congruence theory [4]. A general theory of Abelian algebras and varieties is currently being developed, and except for the congruence modular setting there is much to be uncovered.

2. Tame congruence theory

We now present the fragment of tame congruence theory that will be needed in the subsequent sections. The reader is encouraged to consult [4] for more details.

Definition 2.1. Let $\mathbf{A}$ be a finite algebra.

(1) The clone of $\mathbf{A}$, denoted $\operatorname{Clo}\mathbf{A}$ is the collection of all term operations on $\mathbf{A}$. For each n, $\operatorname{Clo}_n \mathbf{A}$ is the set of all n-ary term operations of $\mathbf{A}$.

(2) A polynomial of $\mathbf{A}$ is a function of the form $t(\bar{x}, a_1, \ldots, a_m)$ for some term operation t and some elements $a_1, \ldots, a_m$ of $\mathbf{A}$. The polynomial clone of $\mathbf{A}$, denoted $\operatorname{Pol}\mathbf{A}$, is the set of all polynomial operations on $\mathbf{A}$. For each n, $\operatorname{Pol}_n \mathbf{A}$ is the set of all n-ary polynomial operations of $\mathbf{A}$.

(3) An algebra $\mathbf{A}'$ with the same universe as the algebra $\mathbf{A}$ is said to be polynomially equivalent to $\mathbf{A}$ if $\operatorname{Pol}\mathbf{A} = \operatorname{Pol}\mathbf{A}'$.

(4) A function $f(x)$ is called idempotent if $f(f(x)) = f(x)$ for all x. Equivalently, f is idempotent if and only if f is the identity function on its range.

(5) A minimal set of the algebra $\mathbf{A}$ is a set of the form $p(A)$, where p is a nonconstant unary polynomial operation of $\mathbf{A}$ such that any other unary polynomial of $\mathbf{A}$ whose range is properly contained in $p(A)$ is constant. We will denote the set of all minimal sets of $\mathbf{A}$ by $\mathrm{Min}(\mathbf{A})$. An algebra $\mathbf{A}$ is called minimal if it is finite and $\mathrm{Min}(\mathbf{A}) = \{A\}$.

The following fundamental theorem dealing with the minimal sets of a finite simple algebra was discovered by McKenzie in [6]. He proved this theorem for a much wider class of algebras which he called tame. Since the proof for the simple case is not so long, we present it here.

Theorem 2.2. *Let $\mathbf{A}$ be a finite simple algebra.*

(i) *If $U \in \mathrm{Min}(\mathbf{A})$ then $U = e(A)$ for some idempotent polynomial e of $\mathbf{A}$.*
(ii) *If U, $V \in \mathrm{Min}(\mathbf{A})$ then there are unary polynomials f and g of $\mathbf{A}$ such that $f(U) = V$, $g(V) = U$, $fg|_V = id|_V$ and $fg|_U = id|_U$.*
(iii) *If $U \in \mathrm{Min}(\mathbf{A})$ and a, $b \in A$ with $a \neq b$, then there is some unary polynomial $f(x)$ of $\mathbf{A}$ such that $f(A) = U$ and $f(a) \neq f(b)$.*
(iv) *If a, $b \in A$ then there is a sequence of minimal sets $U_1, \dots, U_n$ of $\mathbf{A}$ such that $a \in U_1$, $b \in U_n$, and for each $i < n$, $U_i \cap U_{i+1} \neq \varnothing$.*

Proof. Let $U \in \mathrm{Min}(\mathbf{A})$ and let

$$K = \{f(x) \in \mathrm{Pol}_1\, \mathbf{A} \colon f(A) \subseteq U\}.$$

Since K is a right ideal in the monoid of all unary polynomials of $\mathbf{A}$ under composition, it is not hard to show that the following relation is a congruence on $\mathbf{A}$:

$$\theta = \{\langle a, b\rangle \colon f(a) = f(b) \quad \text{for all } f \in K\}.$$

Since $\mathbf{A}$ is simple then θ is either equal to 0_A or 1_A. The fact that U is the range of some nonconstant polynomial of $\mathbf{A}$ rules out the possibility that $\theta = 1_A$, so we have that $\theta = 0_A$. From this we can easily derive part (iii) of this theorem, for if $a \neq b$ in A, then $\langle a, b\rangle \notin \theta$, so there must be some $f \in K$ with $f(A) = U$ and $f(a) \neq f(b)$ as required.

To prove (i) choose $f, g \in K$ and $u, v \in U$ such that $f(A) = U$, $u \neq v$ and $g(u) \neq g(v)$. From the minimality of U it follows that $gf(A) = U$ and so $g(U) = U$. Since A is finite, there is some n such that $g^n(x) = x$ for all $x \in U$. But then setting $e(x) = g^n(x)$ we have that $e(A) = U$ and e is idempotent.

For (ii), let U, $V \in \mathrm{Min}(\mathbf{A})$ and choose $e(x) \in \mathrm{Pol}_1\, \mathbf{A}$ such that e is idempotent and $e(A) = V$. Choose $u, v \in V$ and a unary polynomial $f(x)$ of $\mathbf{A}$ such that $f(A) = U$ and $f(u) \neq f(v)$. By replacing f with the polynomial fe we may assume that in fact $f(V) = U$. Similarly, we can find $g(x)$ in

$\mathrm{Pol}_1\,\mathbf{A}$ with $g(U) = V$. Thus the function $fg|_U$ is a permutation of U and so there is some $n > 0$ with $(fg|_U)^n = id|_U$. Letting $h(x) = g(fg)^{n-1}(x)$, it is not hard to show that $h(U) = V$, $hf|_V = id|_V$ and $fh|_U = id|_U$ as required.

To prove (iv), we leave it to the reader to show that the transitive closure of the following relation is a congruence on $\mathbf{A}$ (and not equal to 0_A):

$$\{\langle a, b\rangle : a = b \text{ or } \{a, b\} \in U \text{ for some } U \in \mathrm{Min}(\mathbf{A})\}.$$

Given any subset U of an algebra $\mathbf{A}$, we can define an algebraic structure on U as follows. Let

$$(\mathrm{Pol}\,\mathbf{A})|_U = \{h|_U \in \mathrm{Pol}\,\mathbf{A} : h(U) \subseteq U\}$$

and let

$$\mathbf{A}|_U = \langle U, (\mathrm{Pol}\,\mathbf{A})|_U\rangle.$$

$\mathbf{A}|_U$ is called the algebra induced on U by $\mathbf{A}$. Note that we have not specified a language for this algebra. For the purposes of the following discussion this is not important. Such an algebra without an indexed set of fundamental operations is called a nonindexed algebra.

The following theorem proved by McKenzie and based on a theorem by Pálfy [9] shows that for a finite algebra, there are essentially only five different possibilities for the kinds of algebraic structures that can be induced on the minimal sets of the algebra. It follows from Definition 2.1 (5) that a finite algebra is minimal if and only if every unary polynomial of the algebra is either constant or a permutation.

Theorem 2.3. *Let* $\mathbf{A}$ *be a finite algebra. Then* $\mathbf{A}$ *is minimal if and only if* $\mathbf{A}$ *is polynomially equivalent to one of the following kinds of algebras*:

(i) *A unary algebra* $\langle A, \Pi\rangle$, *where* $\Pi \subseteq \mathrm{Sym}(A)$;
(ii) *a vector space over some finite field*;
(iii) *a two element Boolean algebra*;
(iv) *a two element lattice*;
(v) *a two element semi-lattice.*

It is easy to show that if U is a minimal set of a finite algebra $\mathbf{A}$, then the induced algebra $\mathbf{A}|_U$ is a minimal algebra, and so by the previous theorem must be polynomially equivalent to one of the algebras in the above list. If we add the assumption that the algebra is Abelian then we can rule out the last three types of algebras to conclude the following.

Corollary 2.4. *Let* $\mathbf{A}$ *be a finite Abelian algebra, and let* $U \in \mathrm{Min}(\mathbf{A})$. *Then* $\mathbf{A}|_U$ *is polynomially equivalent to a unary minimal algebra or a vector space. If the algebra* $\mathbf{A}$ *is also simple, then* $\mathbf{A}|_U$ *is simple and is isomorphic* (*as a nonindexed algebra*) *to* $\mathbf{A}|_V$ *for any other* $V \in \mathrm{Min}(\mathbf{A})$.

Proof. In the two element meet semi-lattice we have $0 \wedge 1 = 0 \wedge 0$ and $1 \wedge 1 \neq 1 \wedge 0$, showing that this algebra is not Abelian. Since this operation is present in the clone of the two element lattice and the two element Boolean algebra, it follows that these algebras are also not Abelian.

From Theorem 2.2, the polynomial bijection between two minimal sets U and V of a simple algebra provides us with an isomorphism of $\mathbf{A}|_U$ and $\mathbf{A}|_V$. The fact that any minimal set U of the simple algebra $\mathbf{A}$ is the range of an idempotent polynomial is enough to show that $\mathbf{A}|_U$ is simple. In fact, given any algebra $\mathbf{B}$ and any subset V of B which is the range of some idempotent polynomial of $\mathbf{B}$, it was shown by Pálfy and Pudlák [10] that the congruence lattice of $\mathbf{B}|_V$ is a homomorphic image of $\operatorname{Con}\mathbf{B}$.

3. Simple Abelian algebras

For this section, let $\mathbf{A}$ be a finite simple Abelian algebra. The terminology used in the following definition is due to Peter Pröhle.

Definition 3.1. (1) Let $p(\bar{x})$ and $q(\bar{x})$ belong to $\operatorname{Pol}(\mathbf{A})$. We say that p and q are twins if there is a term $t(\bar{x},\bar{y})$ of $\mathbf{A}$ and elements $\bar{u}$ and $\bar{v}$ of A such that $p(\bar{x}) = t(\bar{x},\bar{u})$ and $q(\bar{x}) = t(\bar{x},\bar{v})$.

(2) Let $B, C \subseteq A$. We say that B and C are twins if there are twin polynomials $p(x)$ and $q(x)$ of $\mathbf{A}$ such that the range of p is B and the range of q is C.

Lemma 3.2. (i) *Let $p(x)$, $q(x) \in \operatorname{Pol}_1(\mathbf{A})$ be twins. Then p and q have the same kernels, and* $\operatorname{Card}(p(A)) = \operatorname{Card}(q(A))$.

(ii) *Let $U \in \operatorname{Min}(\mathbf{A})$ and $V \subseteq A$ with U and V twins. Then $V \in \operatorname{Min}(\mathbf{A})$ too.*

Proof. Suppose that $p(x)$ and $q(x)$ are twins, say $p(x) = t(x,\bar{c})$ and $q(x) = t(x,\bar{d})$ for some term t of $\mathbf{A}$ and some elements $\bar{c}$ and $\bar{d}$ from A. From the Abelian property we have that for all a, b from $A, p(a) = p(b)$ if and only if $t(a,\bar{c}) = t(b,\bar{c})$ if and only if $t(a,\bar{d}) = t(b,\bar{d})$ if and only if $q(a) = q(b)$. Thus p and q have the same kernel. Since A is finite, it follows that $\operatorname{Card}(p(A)) = \operatorname{Card}(q(A))$. Item (ii) follows from (i) and the definition of $\operatorname{Min}(\mathbf{A})$.

In the main theorem of this paper we essentially reduce our argument to the following proposition.

Proposition 3.3. *Let $\mathbf{A}$ be a finite simple algebra that is either essentially unary, or is polynomially equivalent to a vector space. Then $\mathbf{A}$ is strictly simple.*

Proof. Suppose that $\mathbf{A}$ is essentially unary and $\mathbf{B}$ is a proper subalgebra of $\mathbf{A}$ with more than one element. Let θ be the congruence on $\mathbf{A}$ generated by identifying all of the elements of B. Since $\mathbf{A}$ is essentially unary, no other elements of A get identified under θ. Thus θ is a nontrivial congruence and so $\mathbf{A}$ cannot be simple.

We next examine the case where our algebra $\mathbf{A}$ is assumed to be polynomially equivalent to some vector space $\mathbf{V}$ over some field $\mathbf{F}$. If $\mathbf{B}$ is a nontrivial subalgebra of $\mathbf{A}$, choose some element 0 from B. Using the fact that $\mathbf{A}$ is Abelian, we may assume without loss of generality that this element 0 is the additive zero of our vector space $\mathbf{V}$. It follows that the vector space addition $x + y$ is a polynomial of $\mathbf{A}$ defined from some term of $\mathbf{A}$ using only the parameter 0. An important consequence of this is that the subalgebra $\mathbf{B}$ is closed under +. Similarly, multiplication by any field element λ of $\mathbf{F}$ is also a polynomial operation under which B is closed. So in fact B is a subspace of $\mathbf{V}$. The reader may wish to consult [4], Theorem 4.7 for more details of this argument.

Let θ be the congruence of $\mathbf{A}$ generated by identifying all of the elements of B. We now argue that the set B is a congruence block of θ, thereby showing that $\mathbf{A}$ cannot be simple. If B is not a block of θ, then there must exist a unary polynomial $p(x)$ of $\mathbf{A}$, and elements a, b in B such that $p(a)$ is in B and $p(b)$ is not in B. Since $\mathbf{A}$ is polynomially equivalent to the vector space $\mathbf{V}$, then for some field element λ and some element d from A we have $p(x) = \lambda \cdot x + d$. Now, $p(a) \in B$ implies that $\lambda \cdot a + d$ is in B, and since $\lambda \cdot a$ is also in B then we must have that d is in B too. But then $p(b) = \lambda \cdot b + d$ must be in B, contrary to our assumptions. Thus B is indeed a block of the congruence θ.

We now prove the main result of this paper.

Theorem 3.4. *Let* $\mathbf{A}$ *be a finite simple Abelian algebra. Then* $\mathbf{A}$ *has no nontrivial subalgebras.*

Proof. We argue by contradiction. Let $\mathbf{B}$ be a proper subalgebra of $\mathbf{A}$ containing at least two elements and let $\mathbf{B}$ be maximal with this property. Without loss of generality we may assume that every element $b \in B$ is named by some constant in the language of $\mathbf{A}$. It then follows by the maximality of $\mathbf{B}$ that for every $a \in A \setminus B$ and every $c \in A$ there is a unary term $t(x)$ in the language of $\mathbf{A}$ with $c = t^{\mathbf{A}}(a)$. Also, for every polynomial $p(\bar{x})$ of $\mathbf{A}$, there is some term $r(\bar{x}, y)$ with $p(\bar{x}) = r(\bar{x}, a)$.

From Theorem 2.2 in the previous section we know that every $U \in \mathrm{Min}(\mathbf{A})$ is the range of some idempotent polynomial $e(x)$ on $\mathbf{A}$. Since A is finite we may assume that e arises from a term $t(x, \bar{y})$ which is idempotent in the variable x (i.e. $\mathbf{A} \models t(t(x, \bar{y}), \bar{y}) \approx t(x, \bar{y})$). We will call a minimal set U a B-minimal set of $\mathbf{A}$ if $U = t(A, \bar{b})$ for such a term t and for some elements $\bar{b}$ from B.

Claim. Every B-minimal set of $\mathbf{A}$ is contained entirely within B.

Proof. Let U be a B-minimal set, say $U = t(A, \bar{b})$ with $\bar{b}$ in B and t idempotent in x. Since the elements of $\bar{b}$ are named by some constants in the language, there is a term $s(x)$ with $A \models s(x) = t(x, \bar{b})$. We prove this claim in several steps.

First observe that $U \cap B \neq \varnothing$. Since $s(A) = U$ and s is a term, then B is closed under s, and so $s(B) \subseteq B \cap U$.

Secondly, $\mathrm{Card}(U \cap B) \geq 2$. This follows from the assumption that B contains at least two elements, say $a, b \in B$ with $a \neq b$. By Theorem 2.2, there is a polynomial $f(x)$ on $\mathbf{A}$ with $f(A) \subseteq U$ and $f(a) \neq f(b)$. We may assume that $f(x) = r(x, \bar{c})$ for some term r whose range is contained in U (if not, just apply s on the left) and for some $\bar{c}$ in A. In fact, since $\mathbf{A}$ is Abelian, we may assume that $\bar{c}$ is contained in B, for if we replace $\bar{c}$ by some sequence $\bar{d}$ from B, we still have that $f(a) \neq f(b)$ and $f(A) \subseteq U$. Thus $f(B) \subseteq U \cap B$ and since $f(a) \neq f(b)$ we have that $U \cap B$ contains at least two elements.

Finally, we show that $\mathbf{A}|_U$ is polynomially equivalent to an algebra $\widehat{\mathbf{U}}$ that has the set $U \cap B$ as a subuniverse. Since this algebra is simple and unary or polynomially equivalent to a vector space, it follows that $U \cap B = U$ as claimed. Let

$$S = \{r(x, y) \colon r \text{ is a term of } \mathbf{A} \text{ and } U \text{ is closed under it }\}.$$

If $B \cap U$ is a proper subset of U, choose $a \in U \setminus B$. It then follows from the remarks preceding this claim that

$$\mathrm{Pol}_1(\mathbf{A}|_U) = \{r(x, a)|_U \colon r \in S\}.$$

Define $\widehat{\mathbf{U}}$ to be the following nonindexed algebra:

$$\langle U, \{r(x, y)|_U \colon r \in S\}\rangle.$$

Since the induced algebra $\mathbf{A}|_U$ is minimal and Abelian, $\mathrm{Pol}(\widehat{\mathbf{U}}) \subseteq \mathrm{Pol}(\mathbf{A}|_U)$ and $\mathrm{Pol}_1(\widehat{\mathbf{U}}) = \mathrm{Pol}_1(\mathbf{A}|_U)$, it follows that $\widehat{\mathbf{U}}$ is also minimal and Abelian (in fact, $\mathbf{A}|_U$ and $\widehat{\mathbf{U}}$ are polynomially equivalent). Since the basic operations of $\widehat{\mathbf{U}}$ are merely restrictions of terms of $\mathbf{A}$, and B is a subuniverse of $\mathbf{A}$, then $B \cap U$ is a subuniverse of $\widehat{\mathbf{U}}$. This contradicts the fact mentioned in Proposition 3.3 above. Thus U is contained within B as claimed.

From the simplicity of $\mathbf{A}$ we know that there is a minimal set V with $V \cap B \neq \varnothing$ and $V \setminus B \neq \varnothing$. Choose $a \in V \setminus B$ and $b \in V \cap B$ and a term $t(x, y)$ with t idempotent in the variable x and $t(A, a) = V$.

From the above claim it follows that $t(A, B) \subseteq B$, since $t(A, c) \in \mathrm{Min}(\mathbf{A})$ for all $c \in A$, and if $c \in B$ then the minimal set $t(A, c) \subseteq B$. We also have $t(a, a) = a$ and $t(b, a) = b$ since t is idempotent in x and $a, b \in V = t(A, a)$.

Since A is finite, there is some $k > 0$ with

$$\mathbf{A} \models t_1^{(k)}(x, t_1^{(k)}(x, y)) \approx t_1^{(k)}(x, y),$$

where $t_1^{(k)}(x, y)$ is the term

$$t(x, t(x, \cdots, t(x, y)) \cdots),$$

with t being repeated k times. Let $s(x, y) = t_1^{(k)}(x, y)$. Then we have that $s(a, a) = a$, $s(b, a) \in B$ and $s(A, B) \subseteq B$. From the equation

$$\mathbf{A} \models s(x, s(x, y)) \approx s(x, y)$$

it follows that

$$\mathbf{A} \models s(z, s(x, y)) \approx s(z, y)$$

since $\mathbf{A}$ is Abelian. But $s(a, s(b, a)) \in B$ since $s(b, a) \in B$ and $s(a, a) = a \notin B$, which contradicts the above equation.

4. Conclusion

After stating the next definition, we may rephrase our result from the last section in a different way.

Definition 4.1. An algebra $\mathbf{A}$ is called Hamiltonian if every subalgebra of $\mathbf{A}$ is a block of some congruence on $\mathbf{A}$. A variety is called Hamiltonian if every one of its members is Hamiltonian.

For finite simple algebras, it is not difficult to see that being Hamiltonian is equivalent to being strictly simple, so we have proved that every finite simple Abelian algebra is Hamiltonian. The property of being Hamiltonian can be regarded as another generalization of the notion of a group being commutative. It is not hard to see that every commutative group is indeed Hamiltonian, but there are examples of finite noncommutative groups which are Hamiltonian. This property has been studied by several people, and was introduced in [2] and [11].

In [4] it is asked whether or not every locally finite Abelian variety is Hamiltonian. As noted in the next proposition the converse is true, that is, every Hamiltonian variety is Abelian.

Proposition 4.2. (i) *If* $\mathbf{A}^2$ *is Hamiltonian then* $\mathbf{A}$ *is Abelian.*

(ii) *If* V *is a Hamiltonian variety then it is also Abelian.*

Proof. From the definition of being Abelian, it follows that $\mathbf{A}$ is Abelian if and only if the diagonal subalgebra of $\mathbf{A}^2$ is a block of a congruence on $\mathbf{A}^2$. Thus $\mathbf{A}^2$ Hamiltonian implies that $\mathbf{A}$ is Abelian. Item (ii) follows trivially from (i).

The results in [5] and the theorem proved in this paper provide some evidence that for locally finite varieties the equivalence between the Abelian and Hamiltonian properties holds. By applying some of the techniques used in this paper, along with some results from [8] we have been able to obtain some partial results showing that under certain circumstances, algebras in locally finite Abelian varieties must be Hamiltonian. McKenzie in [7] has modified the arguments contained in this paper to prove that any maximal proper subalgebra of a finite algebra in an Abelian variety is a block of a congruence on the algebra.

Acknowledgment

We would like to thank Emil Kiss, Peter Pröhle, Agnes Szendrei and Gabor Czedli for their hospitality and suggestions while the author was working on this material at the Workshop on Universal Algebra in Budapest during the summer of 1988.

REFERENCES

1. Stanley Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer-Verlag, New York, 1981.
2. B. Csákány, *Abelian properties of primitive classes of universal algebras*, Acta. Sci. Math. (Szeged) **25** 1964, 202–208.
3. Ralph Freese and Ralph McKenzie, *Commutator theory for congruence modular varieties*, Vol. 125, London Math. Soc. Lec. Note Series, Cambridge University Press, 1987.
4. David Hobby and Ralph McKenzie, *The structure of finite algebras*, Vol. 76, Contemp. Math., American Mathematical Society, Providence, Rhode Island, 1988.
5. Emil Kiss and Matthew Valeriote, *Strongly Abelian varieties and the Hamiltonian property* (preprint), 1989.
6. Ralph McKenzie, *Finite forbidden lattices*, in Universal Algebra and Lattice Theory, Vol. 1004, Springer Lecture Notes, Springer-Verlag, New York, 1983.
7. ——, *Congruence extension, Hamiltonian and Abelian properties in locally finite varieties* (preprint), 1989.
8. Ralph McKenzie and Matthew Valeriote, *The structure of locally finite decidable varieties*, Birkhäuser, Boston, 1989.
9. P. P. Pálfy, *Unary polynomials in algebras*, Algebra Universalis, **18** (1984) 262–273.
10. P. P. Pálfy and P. Pudlák, *Congruence lattices of finite algebras and intervals in subgroup lattices of finite groups*, Algebra Universalis **11** (1980), 22–27.
11. K. Shoda, *Zur theorie der algebraischen erweiterungen*, Osaka J. Math. **4** (1952), 133–143.

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCMASTER UNIVERSITY, HAMILTON, ONTARIO, L8S 4K1, CANADA