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# A CHARACTERIZATION OF THE ELEMENTS OF THE SOCLE OF A JORDAN ALGEBRA

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ABSTRACT. Let $J$ be a nondegenerate Jordan algebra over a field $K$ of characteristic not 2. Here we prove that an element $b \in J$ is in the socle if and only if $J$ satisfies dcc on all principal inner ideals $U_yJ$, $y \in Kb + U_bJ$. By using this result we show that the socle of a quadratic extension $J_F$ of $J$ coincides with the quadratic extension $\mathrm{Soc}(J)_F$ of its socle.

Throughout this paper $J$ denotes a (linear) Jordan algebra over a field $K$ of characteristic $\neq 2$. Our standard references for Jordan algebras are [6], [7], [11], For $x, y \in J$ we write their product by $x \cdot y$. For $x, y, z \in J$ we write

(1) $L_x : J \to J \quad L_x y = x \cdot y$
(2) $U_x : J \to J \quad U_x y = 2L_x^2 - L_{x^2}$
(3) $\{xyz\} = (U_{x+z} - U_x - U_z)y$
(4) $B_{x,y} z = z - \{xyz\} + U_x U_y z$.

The Jordan algebra $J$ is said to be *nondegenerate* if $U_x = 0$ implies $x = 0$. An *inner ideal* is a subspace $I$ of $J$ such that $U_I J \subset I$. For any $x, y$ in $J$ we have the *principal* inner ideal $U_x J$, the inner ideal $I(x) = Kx + U_x J$ generated by $x$, and the Bergmann inner ideal $B_{x,y} J$ [7]. For nondegenerate $J$, the *socle* $\mathrm{Soc}(J)$ is defined to be the linear span of all minimal inner ideals of $J$; $\mathrm{Soc}(J) = 0$ if $J$ does not contain any minimal inner ideal. By [10], if $J$ contains minimal inner ideal then $\mathrm{Soc}(J)$ is a direct sum of simple ideals each of which contains a *completely primitive idempotent* $e$ ( $U_e J$ is a division Jordan algebra).

An associative algebra $A$ is semiprime iff the Jordan algebra $A^+$ defined by the product $x \cdot y = \frac{1}{2}(xy + yx)$ is nondegenerate. For semiprime $A$, the (associative) socle of $A$ coincides with the socle of the Jordan algebra $A^+$ (see [3]). It is well known that an element $a \in A$ is in the socle iff $A$ satisfies dcc on all principal left ideals contained in $Aa$. In fact, $A$ satisfies dcc on all left ideals contained in $Aa$ for every $a \in \mathrm{Soc}(A)$. In the workshop on Jordan structures held at the University of Ottawa in 1986, McCrimmon settled the

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following Jordan characterization of the elements in the socle of a semiprime associative algebra $A$: $x \in \operatorname{Soc}(A)$ if and only if $A$ satisfies dcc on all inner ideals contained in $Kx + xAx = Kx + U_x A^+$.

This characterization is not true for a general Jordan algebra: the Jordan algebra $J$ of a nondegenerate symmetric bilinear form on a vector space containing an infinite-dimensional totally isotropic subspace satisfies the dcc on principal inner ideals but does not satisfy the dcc on all inner ideals of $U_1 J = J$ [8, p. 465], yet 1 lies in $\operatorname{Soc}(J) = J$. Nevertheless we do obtain an analogous result if we restrict to the principal dcc.

**Theorem 1.** *Let $J$ be a nondegenerate Jordan algebra. An element $b \in J$ is in the socle iff $J$ satisfies* dcc *on all principal inner ideals $U_y J$, $y \in I(b)$. In particular, every $b \in J$ such that $U_b J$ is finite-dimensional belongs to the socle. This last result was proved in* [5].

*Proof*. Suppose first that $b \in \operatorname{Soc}(J)$. By the structure theorem of the socle, we can reduce the problem to the case that $J$ is a simple Jordan algebra with minimal inner ideals. By Litoff theorem for Jordan algebras [1] we have one of the following possibilities:

(i) $b$ belongs to an inner ideal isomorphic to $M_n(D)^+$ for some natural $n$, where $D$ is a division associative algebra.

(ii) $b$ belongs to an inner ideal isomorphic to the Jordan algebra of symmetric elements of $M_n(D)^+$ with respect to an involution $*: M_n(D) \to M_n(D)$.

(iii) $J$ is isomorphic to the Jordan algebra of a nondegenerate symmetric bilinear form.

(iv) $J$ is isomorphic to the simple exceptional Jordan algebra 27-dimensional over its center.

In any case $J$ satisfies dcc on all principal inner ideals $U_y J$, $y \in I(b)$ by [8].

Conversely, suppose that $J$ satisfies dcc on all principal inner ideals $U_y J$, $y \in I(b)$. Set $\mathfrak{B} = \{U_c J : c \in I(b), c - b \in \operatorname{Soc}(J)\}$. Since $U_b J$ belongs to $\mathfrak{B}$, we can choose $U_c J$ minimal in $\mathfrak{B}$. If $U_c J = 0$ then $c = 0$ by nondegeneracy of $J$, and hence $b \in \operatorname{Soc}(J)$. Suppose to the contrary that $U_c J \neq 0$. Then $U_c J$ contains a minimal inner ideal $I$ of the form $I = U_x J$ where $x = U_x y$ for some $y$ in $J$ [7, p. 106]. For $d = B_{x,y} c$ we have $U_d J \subset U_c J$, but this inclusion is strict since $x$ lies in $U_c J$ but not in $U_d J$ ($B_{x,y} x = 0$ but $B_{x,y}$ is the identity on $U_d J$, $B_{x,y} U_d = B_{x,y} U(B_{x,y} c) = B_{x,y} B_{x,y} U_c B_{y,x} = B_{x,y} U_c B_{y,x} = U_d$ since $(B_{x,y})^2 = B_{x,y}$ by [7, (JP25), p. 21]), yet $U_d J$ is in $\mathfrak{B}$ since $d$ is in $I(c) \subset I(b)$ and $x$ is in $I \subset \operatorname{Soc}(J)$ so that $d \equiv c \equiv b \mod \operatorname{Soc}(J)$. This contradicts the minimality of $U_c J$ in $\mathfrak{B}$.

The following result answers in the affirmative a question proposed in [2].

**Theorem 2.** *Let $J$ be a nondegenerate Jordan algebra and let $F$ be a quadratic extension of the field $K$. Then the scalar extension $J_F = F \otimes_K J$ is also nondegenerate with* $\operatorname{Soc}(J_F) = \operatorname{Soc}(J)_F$.

*Proof*. Without loss in generality we may assume that $F = K(\alpha)$ with $\alpha^2 \in K$ but $\alpha \in F \backslash K$. Then the mapping $a + \alpha b \to a - \alpha b$ is an involution of the Jordan algebra $J_F = J \oplus \alpha J$ over $K$. Hence if $a + \alpha b$ is an absolute zero divisor (a.z.d.) then $a - \alpha b$ is also an a.z.d. By [11, p. 335], $U_{2a}x = U_{(a+\alpha b)+(a-\alpha b)}x$ is an a.z.d. for all $x \in J$, so $a = 0$ by nondegeneracy of $J$. Then $\alpha b$ is an a.z.d. and hence $b = 0$ by nondegeneracy of $J$ again. This proves that $J_F$ is nondegenerate. Let $e$ be a completely primitive idempotent in $J$ and let $M(e)$ denote the simple ideal of $J$ generated by $e$. By [4, Lemma 2.ii], $e$ belongs to $\mathrm{Soc}(J_F)$ and hence $M(e) \subset \mathrm{Soc}(J_F)$. Since $\mathrm{Soc}(J)$ is the sum of all $M(e)$, $\mathrm{Soc}(J)_F \subset \mathrm{Soc}(J_F)$. Conversely, let $x + \alpha y \in \mathrm{Soc}(J_F)$. Since $a + \alpha b \to a - \alpha b$ is a $K$-involution on $J_F$, and the socle is invariant under all ring automorphisms, $x, y \in \mathrm{Soc}(J_F)$. We must show that $x, y \in \mathrm{Soc}(J)$. Suppose to the contrary that $x \notin \mathrm{Soc}(J)$. By Theorem 1 there exists an infinite sequence $\{x_n\} \subset I(x)$ such that the sequence of principal inner ideals $\{U(x_n)J\}$ is strictly descending. Hence $\{U(x_n)J_F\}$ has the same property, which is a contradiction, by Theorem 1 again, because $x \in \mathrm{Soc}(J_F)$.

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