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A CHARACTERIZATION OF THE ELEMENTS OF THE SOCLE OF A JORDAN ALGEBRA

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ABSTRACT. Let J be a nondegenerate Jordan algebra over a field K of characteristic not 2. Here we prove that an element $b \in J$ is in the socle if and only if J satisfies dcc on all principal inner ideals $U_y J$, $y \in Kb + U_b J$. By using this result we show that the socle of a quadratic extension J_F of J coincides with the quadratic extension $\operatorname{Soc}(J)_F$ of its socle.

Throughout this paper J denotes a (linear) Jordan algebra over a field K of characteristic $\neq 2$. Our standard references for Jordan algebras are [6], [7], [11], For $x, y \in J$ we write their product by $x \cdot y$. For $x, y, z \in J$ we write

(1) $L_x : J \to J \quad L_x y = x \cdot y$
(2) $U_x : J \to J \quad U_x y = 2L_x^2 - L_{x^2}$
(3) $\{xyz\} = (U_{x+z} - U_x - U_z)y$
(4) $B_{x,y} z = z - \{xyz\} + U_x U_y z$.

The Jordan algebra J is said to be *nondegenerate* if $U_x = 0$ implies $x = 0$. An *inner ideal* is a subspace I of J such that $U_I J \subset I$. For any x, y in J we have the *principal* inner ideal $U_x J$, the inner ideal $I(x) = Kx + U_x J$ generated by x, and the Bergmann inner ideal $B_{x,y} J$ [7]. For nondegenerate J, the *socle* $\operatorname{Soc}(J)$ is defined to be the linear span of all minimal inner ideals of J; $\operatorname{Soc}(J) = 0$ if J does not contain any minimal inner ideal. By [10], if J contains minimal inner ideal then $\operatorname{Soc}(J)$ is a direct sum of simple ideals each of which contains a *completely primitive idempotent* e ($U_e J$ is a division Jordan algebra).

An associative algebra A is semiprime iff the Jordan algebra A^+ defined by the product $x \cdot y = \frac{1}{2}(xy + yx)$ is nondegenerate. For semiprime A, the (associative) socle of A coincides with the socle of the Jordan algebra A^+ (see [3]). It is well known that an element $a \in A$ is in the socle iff A satisfies dcc on all principal left ideals contained in Aa. In fact, A satisfies dcc on all left ideals contained in Aa for every $a \in \operatorname{Soc}(A)$. In the workshop on Jordan structures held at the University of Ottawa in 1986, McCrimmon settled the

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following Jordan characterization of the elements in the socle of a semiprime associative algebra A: $x \in \operatorname{Soc}(A)$ if and only if A satisfies dcc on all inner ideals contained in $Kx + xAx = Kx + U_x A^+$.

This characterization is not true for a general Jordan algebra: the Jordan algebra J of a nondegenerate symmetric bilinear form on a vector space containing an infinite-dimensional totally isotropic subspace satisfies the dcc on principal inner ideals but does not satisfy the dcc on all inner ideals of $U_1 J = J$ [8, p. 465], yet 1 lies in $\operatorname{Soc}(J) = J$. Nevertheless we do obtain an analogous result if we restrict to the principal dcc.

Theorem 1. *Let J be a nondegenerate Jordan algebra. An element $b \in J$ is in the socle iff J satisfies* dcc *on all principal inner ideals $U_y J$, $y \in I(b)$. In particular, every $b \in J$ such that $U_b J$ is finite-dimensional belongs to the socle. This last result was proved in* [5].

Proof. Suppose first that $b \in \operatorname{Soc}(J)$. By the structure theorem of the socle, we can reduce the problem to the case that J is a simple Jordan algebra with minimal inner ideals. By Litoff theorem for Jordan algebras [1] we have one of the following possibilities:

(i) b belongs to an inner ideal isomorphic to $M_n(D)^+$ for some natural n, where D is a division associative algebra.

(ii) b belongs to an inner ideal isomorphic to the Jordan algebra of symmetric elements of $M_n(D)^+$ with respect to an involution $*: M_n(D) \to M_n(D)$.

(iii) J is isomorphic to the Jordan algebra of a nondegenerate symmetric bilinear form.

(iv) J is isomorphic to the simple exceptional Jordan algebra 27-dimensional over its center.

In any case J satisfies dcc on all principal inner ideals $U_y J$, $y \in I(b)$ by [8].

Conversely, suppose that J satisfies dcc on all principal inner ideals $U_y J$, $y \in I(b)$. Set $\mathfrak{B} = \{U_c J : c \in I(b), c - b \in \operatorname{Soc}(J)\}$. Since $U_b J$ belongs to $\mathfrak{B}$, we can choose $U_c J$ minimal in $\mathfrak{B}$. If $U_c J = 0$ then $c = 0$ by nondegeneracy of J, and hence $b \in \operatorname{Soc}(J)$. Suppose to the contrary that $U_c J \neq 0$. Then $U_c J$ contains a minimal inner ideal I of the form $I = U_x J$ where $x = U_x y$ for some y in J [7, p. 106]. For $d = B_{x,y} c$ we have $U_d J \subset U_c J$, but this inclusion is strict since x lies in $U_c J$ but not in $U_d J$ ($B_{x,y} x = 0$ but $B_{x,y}$ is the identity on $U_d J$, $B_{x,y} U_d = B_{x,y} U(B_{x,y} c) = B_{x,y} B_{x,y} U_c B_{y,x} = B_{x,y} U_c B_{y,x} = U_d$ since $(B_{x,y})^2 = B_{x,y}$ by [7, (JP25), p. 21]), yet $U_d J$ is in $\mathfrak{B}$ since d is in $I(c) \subset I(b)$ and x is in $I \subset \operatorname{Soc}(J)$ so that $d \equiv c \equiv b \mod \operatorname{Soc}(J)$. This contradicts the minimality of $U_c J$ in $\mathfrak{B}$.

The following result answers in the affirmative a question proposed in [2].

Theorem 2. *Let J be a nondegenerate Jordan algebra and let F be a quadratic extension of the field K. Then the scalar extension $J_F = F \otimes_K J$ is also nondegenerate with* $\operatorname{Soc}(J_F) = \operatorname{Soc}(J)_F$.

Proof. Without loss in generality we may assume that $F = K(\alpha)$ with $\alpha^2 \in K$ but $\alpha \in F \backslash K$. Then the mapping $a + \alpha b \to a - \alpha b$ is an involution of the Jordan algebra $J_F = J \oplus \alpha J$ over K. Hence if $a + \alpha b$ is an absolute zero divisor (a.z.d.) then $a - \alpha b$ is also an a.z.d. By [11, p. 335], $U_{2a}x = U_{(a+\alpha b)+(a-\alpha b)}x$ is an a.z.d. for all $x \in J$, so $a = 0$ by nondegeneracy of J. Then αb is an a.z.d. and hence $b = 0$ by nondegeneracy of J again. This proves that J_F is nondegenerate. Let e be a completely primitive idempotent in J and let $M(e)$ denote the simple ideal of J generated by e. By [4, Lemma 2.ii], e belongs to $\mathrm{Soc}(J_F)$ and hence $M(e) \subset \mathrm{Soc}(J_F)$. Since $\mathrm{Soc}(J)$ is the sum of all $M(e)$, $\mathrm{Soc}(J)_F \subset \mathrm{Soc}(J_F)$. Conversely, let $x + \alpha y \in \mathrm{Soc}(J_F)$. Since $a + \alpha b \to a - \alpha b$ is a K-involution on J_F, and the socle is invariant under all ring automorphisms, $x, y \in \mathrm{Soc}(J_F)$. We must show that $x, y \in \mathrm{Soc}(J)$. Suppose to the contrary that $x \notin \mathrm{Soc}(J)$. By Theorem 1 there exists an infinite sequence $\{x_n\} \subset I(x)$ such that the sequence of principal inner ideals $\{U(x_n)J\}$ is strictly descending. Hence $\{U(x_n)J_F\}$ has the same property, which is a contradiction, by Theorem 1 again, because $x \in \mathrm{Soc}(J_F)$.

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References

1. P. N. Anh, *Simple Jordan algebras with minimal inner ideals*, Comm. Algebra **14** (1986), 489–492.
2. M. Benslimane, *Charactérisations des algébres de Jordan Banach non commutatives de capacité finie et caractérisation spectrale des algèbres de Jordan Banach non commutatives modulaires annihilatrices*, Thesis, Université de Rabat, 1987.
3. A. Fernández López, *Modular annihilator Jordan algebras*, Comm. Algebra **13** (1985), 2597–2613.
4. A. Fernández López and A. Rodriguez Palacios, *Primitive noncommutative Jordan algebras with nonzero socle*, Proc. Amer. Math. Soc. **96** (1986), 199–206.
5. ——, *On the socle of a noncommutative Jordan algebra*, Manuscripta Math. **56** (1986), 269–278.
6. N. Jacobson, *Structure and representations of Jordan algebras*, Amer. Math. Soc. Coll. Publ. **39**, Providence, Rhode Island, 1968.
7. O. Loos, *Jordan pairs*, Springer Lecture Notes, Vol. 460, Heidelberg, 1975.
8. K. McCrimmon, *Inner ideals in quadratic Jordan algebras*, Trans. Amer. Math. Soc. **159** (1971), 445–468.
9. K. Meyberg, *Lectures on algebras and triple systems*, Lecture Notes, University of Virginia, Charlottesville, Virginia, 1972.
10. J. M. Osborn and M. L. Racine, *Jordan rings with nonzero socle*, Trans. Amer. Math. Soc. **251** (1979), 375–387.
11. K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov and A. I. Shirshov, *Rings that are nearly associative*, Academic Press, New York, 1982.

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