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YET ANOTHER PROOF OF THE LYAPUNOV CONVEXITY THEOREM

ZVI ARTSTEIN

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ABSTRACT. A new proof is given, of the convexity and compactness of the range of an atomless R^n-valued measure.

Several proofs are available for the theorem of A. A. Lyapunov on the range of a vector measure. (The bibliography given here is not exhaustive.) These proofs reflect both the applicability and the value of the theorem. This paper presents yet another proof, one based on a new, useful argument.

The measure theory we use is standard. Let (Ω, Σ) be a measurable space, and let $\mu = (\mu_1, \ldots, \mu_n)$ be an atomless R^n-valued σ-additive finite measure on it. The range of the restriction of μ to a set T in Σ is

$$R(T) = \{\mu(A) \colon A \subset T,\ A \in \Sigma\}.$$

We denote by $|\mu|$ the scalar measure of total variation of μ. From here on we identify sets which differ by only a set of $|\mu|$-measure zero. Thus $T_1 \subset T_2$ means that $|\mu|(T_1 \backslash T_2) = 0$. We denote by chK the closed convex hull of the set $K \subset R^n$. With this notation the Lyapunov theorem reads $chR(\Omega) = R(\Omega)$. We arrive at it as the conclusion of the following result.

Theorem. *Let x be in $chR(\Omega)$. consider the subclass Σ^1 of Σ, consisting of those $T \in \Sigma$ such that $x \in chR(T)$. Then Σ^1 contains a minimal set, say S, with respect to inclusion (minimal up to $|\mu|$-null set). For the minimal set S we have $\mu(S) = x$. In particular $x \in R(\Omega)$, and the latter is therefore closed and convex.*

We use the following result.

Lemma. *Let $T = \bigcap_{i=1}^{\infty} T_i$, where $T_1 \supset T_2 \supset \cdots$ is a decreasing sequence in Σ. Then $chR(T) = \bigcap_{i=1}^{\infty} chR(T_i)$.*

Proof. The inclusion of $ch(R(T))$ in the intersection is trivial. To verify the other direction, and since all the sets are compact, it suffices to prove that if $y_i \in chR(T_i)$ then the distance between y_i and $ch(R(T))$ tends to zero as

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$i \to \infty$. Since the closure and taking convex hull operations do not increase the distance from the convex set $chR(T)$, it is enough to verify the claim for $y_i \in R(T_i)$, namely when $y_i = \mu(A_i)$ for $A_i \subset T_i$. In particular $y_i = \mu(A_i \cap T) + \mu(A_i \backslash T)$. The first term belongs to $R(T)$; the second term is bounded in norm by $|\mu|(T_i \backslash T)$. The latter sequence converges to zero (an elementary fact of scalar measures, implied by the σ-additivity); hence the vectors $y_i - \mu(A_i \cap T)$ tend to zero and this verifies the claim.

Proof of the existence of a minimal element in Σ^1. Let T_γ, $\gamma \in \Gamma$, be a decreasing family, not necessarily countable, of sets in Σ^1. We claim that a cofinal subsequence T_{γ_i}, $i = 1, 2, \dots$, exists; namely, each T_γ contains an element of the sequence. To show this, consider the numbers $|\mu|(T_\gamma)$, $\gamma \in \Gamma$, and choose a sequence $|\mu|(T_{\gamma_i})$ among these numbers such that each $|\mu|(T_\gamma)$ is greater than or equal to one of the elements in the sequence; T_{γ_i} is then cofinal. Clearly, $T = \bigcap_{i=1}^{\infty} T_{\gamma_i}$ is included (up to $|\mu|$-null sets) in each T_γ. By the lemma, if each T_γ belongs to Σ^1, then $T \in \Sigma^1$; i.e. T_γ, $\gamma \in \Gamma$, has a lower bound in Σ^1. By the Zorn lemma a minimal element exists.

Some notations. Let $p \cdot x$ denote the scalar product of p and x in R^n. If $K \subset R^n$ and $p \in R^n$, then K_p is the p-boundary of K; namely $K_p = \{y \in K : p \cdot y \geq p \cdot z \text{ for all } z \in K\}$. For $K \subset R^n$ and $y \in R^n$, we write $y + K$ for $\{y + z : z \in K\}$. We fix $p \in R^n$. Note that the set function $p \cdot \mu$, defined by $(p \cdot \mu)(A) = p \cdot \mu(A)$, is a σ-additive signed measure. For $T \in \Sigma$ we denote by T_+, T_-, and T_0 the decomposition of T into sets, such that $p \cdot \mu$ is nonnegative on subsets of T_+ and nonpositive on subsets of T_-, and such that $|p \cdot \mu|$ vanishes on T_0 and T_0 is maximal in the sense that $|p \cdot \mu|(A) = 0$ then $|\mu|(A \backslash T_0) = 0$ (namely $|\mu|$ is absolutely continuous with respect to $p \cdot \mu$ on $T_+ \cup T_-$). It is easy to construct this decomposition (e.g. if $f(w)$ is the Radon-Nikodym derivative of μ with respect to $|\mu|$, then $T_+ = \{w \in T : p \cdot f(w) > 0\}$, etc.).

Proposition. *Let* $T \in \Sigma$. *Then* $(chR(T))_p = \mu(T_+) + chR(T_0)$.

Proof. The inclusion $\mu(T_+) + chR(T_0)$ in the p-boundary of $chR(T)$ is trivial. To verify the other direction, let $y \in (chR(T))_p$; we have to show that $y \in \mu(T_+) + chR(T_0)$. Since for bounded sets the closure operation and taking convex-hull operation commute, it is enough to verify the inclusion for y in the closure of $R(T)$; namely when $y = \lim \mu(T_j)$ and $T_j \subset T$. We claim that for $|\mu|(T_+ \backslash T_j)$ and $|\mu|(T_- \cap T_j)$, both converge to zero as $j \to \infty$. This follows immediately from the convergence of $p \cdot \mu(T_j)$ to $p \cdot y = \max\{p \cdot z : z \in chR(T)\}$, and the splitting of T into the positive, negative, and neutral parts with respect to $p \cdot \mu$. Once the convergence to zero of $|\mu|(T_- \cap T_j)$ and $|\mu|(T_+ \backslash T_j)$ is established, we notice that y is also the limit of $\mu(T_+) + \mu(T_0 \cap T_j)$. The latter sequence is in $\mu(T_+) + chR(T_0)$,and this is what we have to show.

Proof of the equality $x = \mu(S)$.

Case 1. x is in the relative interior of $chR(S)$. Since the latter contains the zero vector, it follows that x would also be in $chR(S^1)$ if $|\mu|(S\backslash S^1)$ is small enough. Such an S^1 with $|\mu|(S\backslash S^1) > 0$ is easily constructed by the lack of atoms of $|\mu|$. This contradicts the minimality of S; thus x cannot be in the relative interior of $chR(S)$.

Case 2. x is in the relative boundary of $chR(S)$. Then a $p \in R^n$ exists with $x \in (chR(S))_p$ and $p \cdot x > p \cdot y$ for some $y \in R(S)$. By the proposition, $x - \mu(S_+) \in chR(S_0)$, where S_+ and S_0 are defined with respect to p. Clearly S_0 is a minimal set with this property; otherwise minimality of S is contradicted. The linear dimensionality of $chR(S_0)$ is smaller than that of $chR(S)$; thus an induction argument (or repeating the argument $n-1$ times) completes the proof.

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Department of Theoretical Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel