YET ANOTHER PROOF OF
THE LYAPUNOV CONVEXITY THEOREM

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Abstract. A new proof is given, of the convexity and compactness of the range of an atomless $R^n$-valued measure.

Several proofs are available for the theorem of A. A. Lyapunov on the range of a vector measure. (The bibliography given here is not exhaustive.) These proofs reflect both the applicability and the value of the theorem. This paper presents yet another proof, one based on a new, useful argument.

The measure theory we use is standard. Let $(\Omega, \Sigma)$ be a measurable space, and let $\mu = (\mu_1, \ldots, \mu_n)$ be an atomless $R^n$-valued $\sigma$-additive finite measure on it. The range of the restriction of $\mu$ to a set $T$ in $\Sigma$ is

$$R(T) = \{\mu(A) : A \subset T, A \in \Sigma\}.$$ 

We denote by $|\mu|$ the scalar measure of total variation of $\mu$. From here on we identify sets which differ by only a set of $|\mu|$-measure zero. Thus $T_1 \subset T_2$ means that $|\mu|(T_1 \setminus T_2) = 0$. We denote by $chK$ the closed convex hull of the set $K \subset R^n$. With this notation the Lyapunov theorem reads $chR(\Omega) = R(\Omega)$.

We arrive at it as the conclusion of the following result.

Theorem. Let $x$ be in $chR(\Omega)$. Consider the subclass $\Sigma^1$ of $\Sigma$, consisting of those $T \in \Sigma$ such that $x \in chR(T)$. Then $\Sigma^1$ contains a minimal set, say $S$, with respect to inclusion (minimal up to $|\mu|$-null set). For the minimal set $S$ we have $\mu(S) = x$. In particular $x \in R(\Omega)$, and the latter is therefore closed and convex.

We use the following result.

Lemma. Let $T = \bigcap_{i=1}^\infty T_i$, where $T_1 \supset T_2 \supset \cdots$ is a decreasing sequence in $\Sigma$. Then $chR(T) = \bigcap_{i=1}^\infty chR(T_i)$.

Proof. The inclusion of $ch(R(T))$ in the intersection is trivial. To verify the other direction, and since all the sets are compact, it suffices to prove that if $y_i \in chR(T_i)$ then the distance between $y_i$ and $ch(R(T))$ tends to zero as

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Since the closure and taking convex hull operations do not increase the distance from the convex set \( chR(T) \), it is enough to verify the claim for \( y_i \in R(T_i) \), namely when \( y_i = \mu(A_i) \) for \( A_i \subset T_i \). In particular \( y_i = \mu(A_i \cap T) + \mu(A_i \setminus T) \). The first term belongs to \( R(T) \); the second term is bounded in norm by \( |\mu|(T \setminus T) \). The latter sequence converges to zero (an elementary fact of scalar measures, implied by the \( \sigma \)-additivity); hence the vectors \( y_i - \mu(A_i \cap T) \) tend to zero and this verifies the claim.

**Proof of the existence of a minimal element in \( \Sigma^1 \).** Let \( T_\gamma \), \( \gamma \in \Gamma \), be a decreasing family, not necessarily countable, of sets in \( \Sigma^1 \). We claim that a cofinal subsequence \( T_{\gamma_i} \), \( i = 1, 2, \ldots \), exists; namely, each \( T_{\gamma_i} \) contains an element of the sequence. To show this, consider the numbers \( |\mu|(T_{\gamma_i}) \), \( \gamma \in \Gamma \), and choose a sequence \( |\mu|(T_{\gamma_i}) \) among these numbers such that each \( |\mu|(T_{\gamma_i}) \) is greater than or equal to one of the elements in the sequence; \( T_{\gamma_i} \) is then cofinal. Clearly, \( T = \bigcap_{i=1}^{\infty} T_{\gamma_i} \) is included (up to \( |\mu| \)-null sets) in each \( T_\gamma \). By the lemma, if each \( T_{\gamma_i} \) belongs to \( \Sigma^1 \), then \( T \in \Sigma^1 \); i.e. \( T_\gamma \), \( \gamma \in \Gamma \), has a lower bound in \( \Sigma^1 \). By the Zorn lemma a minimal element exists.

**Some notations.** Let \( p \cdot x \) denote the scalar product of \( p \) and \( x \) in \( R^n \). If \( K \subset R^n \) and \( p \in R^n \), then \( K_p \) is the \( p \)-boundary of \( K \); namely \( K_p = \{ y \in K : p \cdot y \geq p \cdot z \text{ for all } z \in K \} \). For \( K \subset R^n \) and \( y \in R^n \), we write \( y + K \) for \( \{ y + z : z \in K \} \). We fix \( p \in R^n \). Note that the set function \( p \cdot \mu \), defined by \( (p \cdot \mu)(A) = p \cdot \mu(A) \), is a \( \sigma \)-additive signed measure. For \( T \in \Sigma \) we denote by \( T_+ \), \( T_- \), and \( T_0 \) the decomposition of \( T \) into sets, such that \( p \cdot \mu \) is nonnegative on subsets of \( T_+ \) and nonpositive on subsets of \( T_- \), and such that \( |p \cdot \mu| \) vanishes on subsets of \( T_0 \) and \( T_0 \) is maximal in the sense that \( |p \cdot \mu|(A) = 0 \) then \( |\mu|(A \setminus T_0) = 0 \) (namely \( |\mu| \) is absolutely continuous with respect to \( p \cdot \mu \) on \( T_+ \cup T_- \)). It is easy to construct this decomposition (e.g. if \( f(w) \) is the Radon-Nikodym derivative of \( \mu \) with respect to \( |\mu| \), then \( T_+ = \{ w \in T : p \cdot f(w) > 0 \} \), etc.).

**Proposition.** Let \( T \in \Sigma \). Then \( (chR(T))_p = \mu(T_+) + chR(T_0) \).

**Proof.** The inclusion \( \mu(T_+) + chR(T_0) \) in the \( p \)-boundary of \( chR(T) \) is trivial. To verify the other direction, let \( y \in (chR(T))_p \); we have to show that \( y \in \mu(T_+) + chR(T_0) \). Since for bounded sets the closure operation and taking convex-hull operation commute, it is enough to verify the inclusion for \( y \) in the closure of \( R(T) \); namely when \( y = \lim \mu(T_j) \) and \( T_j \subset T \). We claim that for \( |\mu|(T_+ \setminus T_j) \) and \( |\mu|(T_- \cap T_j) \), both converge to zero as \( j \to \infty \). This follows immediately from the convergence of \( p \cdot \mu(T_j) \) to \( p \cdot y = \max \{ p \cdot z : z \in chR(T) \} \), and the splitting of \( T \) into the positive, negative, and neutral parts with respect to \( p \cdot \mu \). Once the convergence to zero of \( |\mu|(T_- \cap T_j) \) and \( |\mu|(T_+ \setminus T_j) \) is established, we notice that \( y \) is also the limit of \( \mu(T_+) + \mu(T_0 \cap T_j) \). The latter sequence is in \( \mu(T_+) + chR(T_0) \), and this is what we have to show.
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Proof of the equality \( x = \mu(S) \).

Case 1. \( x \) is in the relative interior of \( chR(S) \). Since the latter contains the zero vector, it follows that \( x \) would also be in \( chR(S^1) \) if \( |\mu|(S\setminus S^1) \) is small enough. Such an \( S^1 \) with \( |\mu|(S\setminus S^1) > 0 \) is easily constructed by the lack of atoms of \( |\mu| \). This contradicts the minimality of \( S \); thus \( x \) cannot be in the relative interior of \( chR(S) \).

Case 2. \( x \) is in the relative boundary of \( chR(S) \). Then a \( p \in \mathbb{R}^n \) exists with \( x \in (chR(S))_p \) and \( p \cdot x > p \cdot y \) for some \( y \in R(S) \). By the proposition, \( x - \mu(S_+) \in chR(S_0) \), where \( S_+ \) and \( S_0 \) are defined with respect to \( p \). Clearly \( S_0 \) is a minimal set with this property; otherwise minimality of \( S \) is contradicted. The linear dimensionality of \( chR(S_0) \) is smaller than that of \( chR(S) \); thus an induction argument (or repeating the argument \( n - 1 \) times) completes the proof.

Bibliography


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