PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 1, January 1990

NOTE ON "THE LOGICALLY SIMPLEST FORM OF THE INFINITY AXIOM"

FRANCO PARLAMENTO AND ALBERTO POLICRITI

(Communicated by Andreas R. Blass)

In [1] it has been proved that if two sets a and b satisfy the following formula φ:

$$\begin{gathered} a \neq b \wedge a \notin b \wedge b \notin a \wedge \\ (\forall x \in a)(\forall u \in x)(u \in b) \wedge (\forall x \in b)(\forall u \in x)(u \in a) \wedge (\forall x \in a)(x \notin b) \wedge \\ (\forall x, y \in a)(\forall z, w \in b)(z \in x \wedge x \in w \wedge w \in y \rightarrow z \in y) \wedge \\ (\forall x, y \in b)(\forall z, w \in a)(z \in x \wedge x \in w \wedge w \in y \rightarrow z \in y), \end{gathered}$$

then either $\omega' \subseteq a$ and $\omega'' \subseteq b$, or $\omega' \subseteq b$ and $\omega'' \subseteq a$, where $\omega' = \{f_n : n \in \omega\}$, $\omega'' = \{g_n : n \in \omega\}$, $f_0 = \varnothing$, $g_n = \{f_0, \dots, f_n\}$, and $f_{n+1} = \{g_0, \dots, g_n\}$.

Since φ is satisfied by ω' and ω'', φ is an example of a restricted purely universal formula which is satisfiable but not finitely satisfiable.

On the other hand, the conditions

(a) $a \neq b$,
(b) $(\forall x \in a)(x \notin b)$,
(c) $(\forall x, y \in a)(\forall z, w \in b)(z \in x \wedge x \in w \wedge w \in y \rightarrow z \in y)$,
(d) $(\forall x, y \in b)(\forall z, w \in a)(z \in x \wedge x \in w \wedge w \in y \rightarrow z \in y)$,

are implied, assuming the axiom of foundation, by the following formula ψ:

$$\begin{gathered} a \neq \varnothing \wedge b \neq \varnothing \wedge a \notin b \wedge b \notin a \wedge \\ (\forall x \in a)(\forall u \in x)(u \in b) \wedge (\forall x \in b)(\forall u \in x)(u \in a) \wedge \\ (\forall x \in a)(\forall y \in b)(x \in y \vee y \in x). \end{gathered}$$

Therefore the formula $(\exists a, b)\psi(a, b)$ also provides a formulation of the infinity axiom. Note, however, that the two formulae are *not* satisfied by the same pairs of objects; for example, the pair $\{\omega'\} \cup \omega'$, $\{\omega''\} \cup \omega''$ satisfies φ but certainly does not satisfy ψ, since neither $\omega' \in \omega''$ nor $\omega'' \in \omega'$ holds, and the last conjunct in ψ is not satisfied.

Acknowledgment

We wish to thank A. Ferro for the previous observation.

Received by the editors January 30, 1989, and in revised form May 1, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 03E30; Secondary 03B25.

References

1. F. Parlamento and A. Policriti, *The logically simplest form of the infinity axiom*, Proc. Amer. Math. Soc. **103** (1988).

Current address (Franco Parlamento): Dipartimento di Matematica e Informatica, Universitá di Udine, 31100 Udine, Italy

Current address (Alberto Policriti): Department of Computer Science, Courant Institute of Mathematical Sciences, New York University, New York, New York, 10012