

MIXED AND DIRECTIONAL DERIVATIVES

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ABSTRACT. The estimate

$$\left\| \frac{\partial^k f(x)}{\partial \xi_1 \cdots \partial \xi_k} \right\| \leq \sup_{\xi} \left\| \frac{\partial^k f(x)}{\partial \xi^k} \right\|$$

is proved for various spaces of functions over domains in R^d , where $\partial f / \partial \xi_i$ is the directional derivative of f in the ξ_i direction and ξ_1, \dots, ξ_k are any k directions.

1. INTRODUCTION

Estimates of mixed derivatives by directional derivatives were proved (implicitly) in the effort to characterize the K -functional of the pair (L_p, W_p^r) (see [1, Ch. 5]). Though this fact is not stated explicitly, the Kemperman Lemma (see [1, Lemma 4.11, p. 338]) actually implies for a domain D which is nice enough

$$(1.1) \quad \left\| \frac{\partial^k f}{\partial \xi_1 \cdots \partial \xi_k} \right\|_{L_p(D)} \leq C(k) \sup_{\xi} \left\| \frac{\partial^k f}{\partial \xi^k} \right\|_{L_p(D)}$$

with $C(k)$ that increases geometrically with k (see [1, Ch. 5]). In this paper we will show that $C(k)$ can be replaced by 1, which is obviously the best possible constant, as the directions ξ_1, \dots, ξ_k are any k directions, or unit vectors, in R^d . We remark also that here the assumption that D is open in R^d is sufficient. Furthermore, the result is valid for many other spaces, as will be shown in §5. We hope that this inequality will help in settling other problems, for example the investigation of the still open problem of best constant for the multivariate Landau-Kolmogorov inequality (see [2]). We further believe that (1.1) with the elegant best constant $C(k) = 1$ is desirable by itself.

2. THE LOCAL RESULT

The local version of the result of this paper is given in the following theorem, which is essentially the basis for all other results of the paper.

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Theorem 2.1. *Suppose $(\partial^k/\partial\xi^k)f(x)$ exists and is continuous in a neighbourhood $U \subset \mathbb{R}^d$ of x_0 for any direction ξ . Then the mixed derivative $(\partial^k/(\partial\xi_1 \cdots \partial\xi_k))f(x)$ exists, is continuous in a neighbourhood of x_0 , and*

$$(2.1) \quad \left| \frac{\partial}{\partial\xi_1} \cdots \frac{\partial}{\partial\xi_k} f(x_0) \right| \leq \sup_{\xi} \left| \frac{\partial^k}{\partial\xi^k} f(x_0) \right|.$$

Remark. We note that ξ_i are any k directions in \mathbb{R}^d .

We observe that Theorem 2.1 implies for an open domain D and $(\partial^k/\partial\xi^k)f(x) \in C(D)$ that

$$\frac{\partial}{\partial\xi_1} \cdots \frac{\partial}{\partial\xi_k} f(x) \in C(D)$$

and

$$(2.2) \quad \left\| \frac{\partial}{\partial\xi_1} \cdots \frac{\partial}{\partial\xi_k} f(x) \right\|_{C(D)} \leq \sup_{\xi} \left\| \left(\frac{\partial}{\partial\xi} \right)^k f(x) \right\|_{C(D)}.$$

We will actually prove the result for the spaces of functions $L_p(D)$ and $C(D)$ simultaneously.

Theorem 2.2. *Suppose $f \in B$, $(\partial^k/\partial\xi^k)f \in B$ for all ξ and $\|\partial^k f/\partial\xi^k\|_B \leq M$, where either $B = L_p(D)$ $1 \leq p < \infty$ or $B = C(D)$, D being an open set in \mathbb{R}^d . Then*

$$\frac{\partial}{\partial\xi_1} \cdots \frac{\partial}{\partial\xi_k} f \in B$$

and

$$(2.3) \quad \left\| \frac{\partial}{\partial\xi_1} \cdots \frac{\partial}{\partial\xi_k} f(x) \right\|_B \leq \sup_{\xi} \left\| \left(\frac{\partial}{\partial\xi} \right)^k f(x) \right\|_B.$$

Remark 2.3. It sufficient to prove that

$$\frac{\partial}{\partial\xi_1} \cdots \frac{\partial}{\partial\xi_k} f \in B_\varepsilon$$

and

$$(2.4) \quad \left\| \frac{\partial}{\partial\xi_1} \cdots \frac{\partial}{\partial\xi_k} f(x) \right\|_{B_\varepsilon} \leq \sup_{\xi} \left\| \frac{\partial^k}{\partial\xi^k} f(x) \right\|_B$$

where $B_\varepsilon = L_p(D_\varepsilon)$ or $B_\varepsilon = C(D_\varepsilon)$ and where

$$(2.5) \quad D_\varepsilon \equiv \{x; x + B(\varepsilon) \subset D\}, \quad B(\varepsilon) \equiv \{y; \|y\| \leq \varepsilon\}$$

and $\|y\|$ is the Euclidean norm in \mathbb{R}^d . That the inequality (2.3) implies (2.4) is clear for $B = C(D)$ and follows from the theorem on monotone convergence for $B = L_p(D)$.

For the proof which will be carried through in §4 using several lemmas stated and proved in §3, we need the following concepts. For a function space on domain D , $S(D)$ we define the transformations

$$(2.6) \quad T(y)f(x) = f(x + y), \quad T(y): S(D) \rightarrow S(D - y)$$

and

$$(2.7) \quad \Delta_y f(x) = (T(y) - I)f(x) = f(x + y) - f(x), \quad \Delta_y: S(D) \rightarrow S(D \cap (D - y)).$$

In the next section, and when proving Theorem 2.2, $S(D)$ will be $L_p(D_*)$ or $C(D_*)$ where D_* would be D_ε and its translates. We give (2.6) and (2.7) in the present generality to accommodate some remarks in §5.

3. SOME CRUCIAL LEMMAS

We first prove the following equivalence result.

Lemma 3.1. *Suppose $f \in L_p(D)$ or $f \in C(D)$. Then the derivative*

$$\frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f(x)$$

exists in $L_p(D)$ or $C(D)$ as a local strong derivative if and only if

$$(3.1) \quad h^{-k} \Delta_{h\xi_1} \cdots \Delta_{h\xi_k} f(x) \rightarrow g(x) \quad \text{in } L_p(D_\varepsilon) \text{ or } C(D_\varepsilon)$$

for every $\varepsilon > 0$. In this case

$$g(x) = \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f(x)$$

a.e. in D_ε for $B = L_p(D)$ and everywhere in D_ε for $B = C(D)$.

Remark. For h small enough, which depends on ε , $\Delta_{h\xi_1} \cdots \Delta_{h\xi_k}$ is defined on $L_p(D_\varepsilon)$.

Proof. By the repeated application of the mean value theorem, we see that the first assumption implies the second for the space of continuous functions. For the space L_p the same argument is used essentially, and we write for $|h| \leq \varepsilon/k$, $x \in D_\varepsilon$

$$(3.2) \quad \begin{aligned} & h^{-k} \Delta_{h\xi_1} \cdots \Delta_{h\xi_k} f(x) \\ &= h^{-k} \int_0^h T(u_k \xi_k) \int_0^h \cdots \int_0^h T(u_1 \xi_1) \frac{\partial^k}{\partial \xi_1 \cdots \partial \xi_k} f(x) du_1 \cdots du_k \text{ a.e.} \end{aligned}$$

Obviously, the operator $O_h: L_p(D) \rightarrow L_p(D_\varepsilon)$ given by

$$O_h \equiv h^{-k} \int_0^h T(u_k \xi_k) \int_0^h \cdots \int_0^h T(u_1 \xi_1) du_1 \cdots du_k$$

is bounded, and

$$\|O_h \varphi - \varphi\|_{L_p(D_\varepsilon)} \rightarrow 0 \quad \text{for all } \varphi \in L_p(D).$$

We now show that for $g(x)$ given by (3.1) and $|t_i| < \varepsilon/k$

$$(3.3) \quad \frac{1}{t_1 \cdots t_k} \Delta_{t_1 \xi_1} \cdots \Delta_{t_k \xi_k} f(x) = \frac{1}{t_1 \cdots t_k} \int_0^{t_k} T(u_k \xi_k) \int_0^{t_{k-1}} \cdots \int_0^{t_1} T(u_1 \xi_1) g(x) du_1 \cdots du_k$$

a.e. in $D_{2\varepsilon}$ when g in (3.1) is in $L_p(D_\varepsilon)$ $1 \leq p < \infty$, and everywhere in $D_{2\varepsilon}$ when $g \in C(D_\varepsilon)$. To show (3.3) we observe that for $|t_i| < \varepsilon/k$

$$O_{t_1 \cdots t_k} \equiv \frac{1}{t_1 \cdots t_k} \int_0^{t_k} T(u_k \xi_k) \cdots \int_0^{t_1} T(u_1 \xi_1) du_1 \cdots du_k$$

is a bounded transformation from $L_p(D_\varepsilon)$ to $L_p(D_{2\varepsilon})$ ($1 \leq p < \infty$) or from $C(D_\varepsilon)$ to $C(D_{2\varepsilon})$. In fact for t_i small enough

$$(3.4) \quad \|O_{t_1 \cdots t_k} \varphi\|_{B_\varepsilon} \leq \|\varphi\|_B$$

and

$$(3.5) \quad \lim_{t_i \rightarrow 0} \|O_{t_1 \cdots t_k} \varphi(x) - \varphi(x)\|_{B_\varepsilon} = 0,$$

no matter what order of t_i we use in the limit. This implies

$$\begin{aligned} O_{t_1 \cdots t_k} g(x) &= \lim_{h \rightarrow 0} h^{-k} O_{t_1 \cdots t_k} \Delta_{h \xi_1} \cdots \Delta_{h \xi_k} f(x) \\ &= \lim_{h \rightarrow 0} O_{h, \dots, h} \left\{ \frac{1}{t_1 \cdots t_k} \Delta_{t_1 \xi_1} \cdots \Delta_{t_k \xi_k} f(x) \right\} \\ &= \frac{1}{t_1 \cdots t_k} \Delta_{t_1 \xi_1} \cdots \Delta_{t_k \xi_k} f(x) \end{aligned}$$

for $x \in D_{2\varepsilon}$. We now write

$$\begin{aligned} g(x) &= \lim_{t_1 \rightarrow 0} \cdots \lim_{t_k \rightarrow 0} O_{t_1 \cdots t_k} g(x) \\ &= \lim_{t_1 \rightarrow 0} \cdots \lim_{t_k \rightarrow 0} \frac{1}{t_1 \cdots t_k} \Delta_{t_1 \xi_1} \cdots \Delta_{t_k \xi_k} f(x) \\ &= \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f(x) \end{aligned}$$

in $L_p(D_{2\varepsilon})$ or in $C(D_{2\varepsilon})$, which implies our result. \square

Lemma 3.2. *Theorem 2.2 is valid with a constant $C(k)$ instead of 1, that is*

$$(3.6) \quad \left\| \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f \right\|_B \leq C(k) \sup_\xi \left\| \left(\frac{\partial}{\partial \xi} \right)^k f(x) \right\|_B.$$

Proof. We have to prove the analogue of (2.4) with a constant $C(k)$ that does not depend on ε . This follows the Kemperman lemma ([1, p. 338]) which yields

$$(3.7) \quad \prod_{i=1}^k \Delta_{h \xi_i} = \sum_{S \subset \{1, \dots, k\}} (-1)^{|S|} T(h_S^*) \Delta_{h_S}^k$$

where the sum is on all the subsets S of $\{1, \dots, k\}$, $|S|$ is the number of elements in S

$$h_S^* \equiv h \sum_{j \in S} \xi_j \quad \text{and} \quad h_S \equiv h \sum_{j \in S} j^{-1} \xi_j.$$

We choose h such that $2kh < \varepsilon$ so that the transformation on the right and on the left of (3.5) is from B to B_ε . The constant can now be estimated, but there is no use for it as eventually the constant 1 will be achieved. \square

Lemma 3.3. *Under the assumptions of Theorem 2.2,*

$$\left\| \frac{\partial^k}{\partial \xi_1 \cdots \partial \xi_k} f \right\|_B$$

is continuous in ξ_j .

Proof. For our purposes it is probably sufficient to prove that

$$\left\| \frac{\partial^k}{\partial \xi_1 \cdots \partial \xi_k} f \right\|_{B_\varepsilon}$$

is continuous in ξ_i , but the result is valid as stated and may be useful in the future. We can assume the change is in ξ_1 . Let $\xi_1^* - \xi_1 = \delta \eta$ where η is a unit vector. By earlier consideration

$$\left\| \frac{\partial}{\partial \xi} \cdots \frac{\partial}{\partial \xi_k} f(x) \right\|_B - \|h^{-k} \Delta_{h\xi} \Delta_{h\xi_2} \Delta_{h\xi_k} f(x)\|_{B_\varepsilon} < \frac{\varepsilon_1}{3}$$

for $\xi = \xi_1$ and $\xi = \xi_1^*$ for ε and h small enough. Using (3.4), we now write

$$\begin{aligned} & \|h^{-k} \Delta_{h\xi_1} \cdots \Delta_{h\xi_k} f(x)\|_{B_\varepsilon} - \|h^{-k} \Delta_{h\xi_1^*} \Delta_{h\xi_2} \cdots \Delta_{h\xi_k} f(x)\|_{B_\varepsilon} \\ & \leq \left\| \delta \frac{1}{h\delta} \frac{1}{h^{k-1}} \Delta_{h\delta\eta} \Delta_{h\xi_2} \cdots \Delta_{h\xi_k} f(x) \right\|_{B_\varepsilon} \\ & \leq \delta \left\| O_{\delta h, h, \dots, h} \frac{\partial}{\partial \eta} \frac{\partial}{\partial \xi_2} \cdots \frac{\partial}{\partial \xi_k} f \right\|_{B_\varepsilon} \\ & \leq \delta \left\| \frac{\partial}{\partial \eta} \frac{\partial}{\partial \xi_2} \cdots \frac{\partial}{\partial \xi_k} f \right\|_B \\ & \leq \delta C(k) \sup \left\| \frac{\partial^k f}{\partial \xi^k} \right\|_B. \end{aligned}$$

The last inequality follows from Lemma 3.2. The result is easily concluded from the above estimate. \square

4. PROOF OF THE MAIN RESULT

We are now able to prove the main result, i.e. Theorem 2.2

Proof of Theorem 2.2. The proof follows by induction on k . For $k = 2$ we use the identity

$$(4.1) \quad \Delta_{h\xi} \Delta_{h\eta} = \Delta_{h(\xi+\eta)/2}^2 - T(h\eta) \Delta_{h(\xi-\eta)/2}^2,$$

set (for $\xi \neq \eta$) $\xi_1 = (\xi + \eta)/\|\xi + \eta\| = (\xi + \eta)/2a$, $\eta_1 = (\xi - \eta)/\|\xi - \eta\| = (\xi - \eta)/2b$ (where the norms are the Euclidean norm of R^d), and recall

$$(4.2) \quad a^2 + b^2 = \left\| \frac{\xi + \eta}{2} \right\|^2 + \left\| \frac{\xi - \eta}{2} \right\|^2 = 2 \left(\left\| \frac{\xi}{2} \right\|^2 + \left\| \frac{\eta}{2} \right\|^2 \right) = 1.$$

Using Lemma 3.1, (4.1), (4.2) and $h < \varepsilon/4$, we have

$$\begin{aligned} \left\| \frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta} f(x) \right\|_{B_\varepsilon} &= \lim_{h \rightarrow 0} \|h^{-2} \Delta_{h\xi} \Delta_{h\eta} f\|_{B_\varepsilon} \\ &\leq \lim_{h \rightarrow 0} \{a^2 \|h^{-2} \Delta_{h\xi_1}^2 f\|_{B_\varepsilon} + b^2 \|h^{-2} \Delta_{h\eta_1}^2 f\|_{B_{\varepsilon/2}}\} \\ &\leq \left\{ a^2 \left\| \frac{\partial^2 f}{\partial \xi_1^2} \right\|_B + b^2 \left\| \frac{\partial^2 f}{\partial \eta_1^2} \right\|_B \right\} \\ &\leq \sup_{\xi} \left\| \frac{\partial^2 f}{\partial \xi^2} \right\|_B, \end{aligned}$$

and therefore,

$$(4.3) \quad \left\| \frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta} f \right\|_B \leq \left\{ a^2 \left\| \frac{\partial^2 f}{\partial \xi_1^2} \right\|_B + b^2 \left\| \frac{\partial^2 f}{\partial \eta_1^2} \right\|_B \right\} \leq \sup_{\xi} \left\| \frac{\partial^2 f}{\partial \xi^2} \right\|_B.$$

We proceed with the induction hypothesis on $0 < m < k$ to obtain

$$\begin{aligned} I &\equiv \sup_{\xi_1, \dots, \xi_k} \left\| \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f \right\|_B \\ &\leq \sup_{\xi, \xi_{m+1}, \dots, \xi_k} \left\| \left(\frac{\partial}{\partial \xi} \right)^m \frac{\partial}{\partial \xi_{m+1}} \cdots \frac{\partial}{\partial \xi_k} f \right\|_B \\ &\leq \sup_{\xi, \eta} \left\| \left(\frac{\partial}{\partial \xi} \right)^m \left(\frac{\partial}{\partial \eta} \right)^{k-m} f \right\|_B \\ &\equiv I(m) \leq I. \end{aligned}$$

We observe, using Lemma 3.3, that maximum can replace supremum in both $I(m)$ and I . We need to show that for some direction ζ

$$I = I(m) = \left\| \frac{\partial^k}{\partial \zeta^k} f \right\|_B.$$

Given $0 < m < k$, we have for some ξ and η

$$I = I(m) = \left\| \frac{\partial^m}{\partial \xi^m} \frac{\partial^{k-m}}{\partial \eta^{k-m}} f \right\|_B.$$

If $\xi = \eta$, the theorem is proved, and therefore, we assume $\xi \neq \eta$. If k is even, we choose m so that $2m = k$, and using the first inequality in (4.3) repeatedly,

we have

$$\begin{aligned} I &= I\left(\frac{k}{2}\right) = \left\| \frac{\partial^m}{\partial \xi^m} \frac{\partial^m}{\partial \eta^m} f \right\|_B \\ &\leq \sum_{l=0}^m \binom{m}{l} a^{2l} b^{2m-2l} \left\| \frac{\partial^{2l}}{\partial \xi_1^{2l}} \frac{\partial^{2m-2l}}{\partial \eta_1^{2m-2l}} f \right\|_B \\ &\leq a^{2m} \left\| \frac{\partial^{2m}}{\partial \xi_1^{2m}} f \right\|_B + \left(\sum_{l=0}^{m-1} \binom{m}{l} a^{2l} b^{2m-2l} \right) I, \end{aligned}$$

and as $a^2 + b^2 = 1$, we have

$$\sum_{l=0}^{m-1} \binom{m}{l} a^{2l} b^{2m-2l} = 1 - a^{2m},$$

and therefore,

$$I = \left\| \frac{\partial^{2m}}{\partial \xi_1^{2m}} f \right\|_B.$$

For odd k we construct a sequence $\{m_i, \xi_i, \eta_i\}$ such that $0 < m_i < k$,

$$(4.4) \quad I = I(m_i) = \left\| \frac{\partial^{m_i}}{\partial \xi_i^{m_i}} \frac{\partial^{k-m_i}}{\partial \eta_i^{k-m_i}} f \right\|_B$$

and the angle between ξ_{i+1} and η_{i+1} is half the angle between ξ_i and η_i . For $m_i < k/2$ we choose $\xi_{i+1} = (\xi_i + \eta_i)/\|\xi_i + \eta_i\|$, $\eta_{i+1} = \eta_i$ and $m_{i+1} = 2m_i$, while for $m_i > k/2$ we choose $\xi_{i+1} = \xi_i$, $\eta_{i+1} = (\xi_i + \eta_i)/\|\xi_i + \eta_i\|$ and $k - m_{i+1} = 2(k - m_i)$. To show that (4.4) is valid for $i + 1$, we write for $m_i < k/2$

$$\zeta \equiv (\xi_i - \eta_i)/\|\xi_i - \eta_i\| \quad \text{and} \quad a_i \equiv \frac{1}{2}\|\xi_i + \eta_i\|,$$

and hence

$$\begin{aligned} I &= I(m_i) \leq \sum_{l=0}^{m_i} \binom{m_i}{l} a_i^{2l} (1 - a_i^2)^{m_i-l} \left\| \frac{\partial^{2l}}{\partial \zeta_i^{2l}} \frac{\partial^{2m_i-2l}}{\partial \zeta_i^{2m_i-2l}} \frac{\partial^{k-2m_i}}{\partial \eta_{i+1}^{k-2m_i}} f \right\|_B \\ &\leq a_i^{2m_i} \left\| \frac{\partial^{2m_i}}{\partial \zeta_i^{2m_i}} \frac{\partial^{k-2m_i}}{\partial \eta_{i+1}^{k-2m_i}} f \right\|_B + (1 - a_i^{2m_i}) I. \end{aligned}$$

The case $m_i > k/2$ is symmetric. As $\|\xi_i - \eta_i\| \rightarrow 0$, Lemma 3.3 yields our theorem. \square

5. EXTENSIONS TO OTHER FUNCTION SPACES AND OTHER DOMAINS

As Theorem 2.2 is valid for $C(D)$ for all open sets D , it is valid for $C(R^d)$. From this we may deduce the result to a Banach space of functions or generalized functions on R^d for which translation is an isometry, that is

$$(5.1) \quad \|T(y)f(\cdot)\|_B = \|f(\cdot + y)\|_B = \|f(\cdot)\|_B.$$

Translation is called strongly, weakly or weakly* continuous if $T(y)f - f \rightarrow 0$ (as $\|y\| \rightarrow 0$) in the strong, weak or weak* topology, respectively (the last only when $X^* = B$, for some Banach space X).

Theorem 5.1. *Suppose B is a Banach space of functions or generalized functions on R^d for which translation is a strongly, weakly or weakly* continuous isometry. Then $\partial^k f / \partial \xi^k \in B$ for all ξ implies $\partial^k f / (\partial \xi_1 \cdots \partial \xi_k) \in B$ where derivatives are taken in the strong, weak or weak* sense, respectively, and*

$$(5.2) \quad \left\| \frac{\partial^k}{\partial \xi_1 \cdots \partial \xi_k} f(\cdot) \right\|_B \leq \sup_{\xi} \left\| \frac{\partial^k}{\partial \xi^k} f(\cdot) \right\|_B.$$

Proof. We define

$$F(x) = \langle f(x + \cdot), g(\cdot) \rangle$$

for $g \in B^*$ in case the strong or weak result is proved, and for $g \in X$, $X^* = B$ in case the weak* result is proved. We set $\|g\|_{B^*} = 1$ or $\|g\|_X = 1$, respectively, and write

$$\left\| \frac{\partial^k F(\cdot)}{\partial \xi_1 \cdots \partial \xi_k} \right\|_{C(R^d)} \leq \sup_{\xi} \left\| \frac{\partial^k F(\cdot)}{\partial \xi^k} \right\|_{C(R^d)} \leq \sup_{\xi} \left\| \frac{\partial^k f(\cdot)}{\partial \xi^k} \right\|_B.$$

As

$$\frac{\partial^k}{\partial \xi_1 \cdots \partial \xi_k} F(0)$$

is given for any g ,

$$\frac{\partial^k f}{\partial \xi_1 \cdots \partial \xi_k}$$

exists and we may choose g_ε such that

$$\begin{aligned} \left\| \frac{\partial^k}{\partial \xi_1 \cdots \partial \xi_k} F \right\|_{C(R^d)} &\geq \left\| \frac{\partial^k}{\partial \xi_1 \cdots \partial \xi_k} F(0) \right\| \\ &= \left| \left\langle \frac{\partial^k}{\partial \xi_1 \cdots \partial \xi_k} f(\cdot), g_\varepsilon(\cdot) \right\rangle \right| \\ &\geq \left\| \frac{\partial^k}{\partial \xi_1 \cdots \partial \xi_k} f \right\|_B - \varepsilon \end{aligned}$$

with any $\varepsilon > 0$. \square

Remark 5.2. Theorem 5.1 applies to $L_\infty(R^d)$ as translation is weakly* continuous isometry in that space.

Remark 5.3. For an open set Ω for which a function $f \in W_p^r(\Omega)$ can be extended to $F \in W_p^r(R^d)$, that is $F(x) = f(x)$ for $x \in \Omega$ and for which $m(\partial\Omega) = 0$, we have

$$(5.3) \quad \left\| \frac{\partial^k}{\partial \xi_1 \cdots \partial \xi_k} F \right\|_{L_p(\bar{\Omega})} \leq \sup_{\xi} \left\| \frac{\partial^k f}{\partial \xi^k} \right\|_{L_p(\Omega)}, \quad 0 < k \leq r.$$

The same is valid for the norms $C(\overline{\Omega})$ and $C(\Omega)$ if Ω is such that $f \in C^r(\Omega)$ can be extended to $F \in C^r(\mathbb{R}^d)$. Such extensions are discussed extensively elsewhere (see [3] and [4]).

Remark 5.4. In Theorem 2.2, $L_p(\Omega)$ can be replaced by any Banach space of functions on Ω , $B(\Omega)$ which satisfy:

$$(1) \quad |f(x)| \geq |g(x)| \text{ for } x \in \Omega \text{ implies } \|f\|_{B(\Omega)} \geq \|g\|_{B(\Omega)},$$

$$(2) \quad \|O_{t_1 \dots t_k} f\|_{B(\Omega_\epsilon)} \leq \|f\|_{B(\Omega)} \text{ for } t_i \text{ small enough}$$

and

$$(3) \quad \|f\|_{B(\Omega_\epsilon)} \leq M \text{ implies } \lim_{\epsilon \rightarrow 0^+} \|f\|_{B(\Omega_\epsilon)} = \|f\|_{B(\Omega)}.$$

It is easy to see that properties (1), (2) and (3) are satisfied by many spaces and are not particular to L_p .

Remark 5.5. As $(\mathbb{R}_+^d)^0 = \Omega$ satisfies the condition in Remark 5.3, (5.3) is valid for $C(\mathbb{R}_+^d)$, and therefore, the method of Theorem 5.1 will imply validity for any Banach spaces on \mathbb{R}_+^d for which translation $T(y)$ for $y \in \mathbb{R}_+^d$ is a contraction which is strongly, weakly or weakly* continuous.

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