PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 1, January 1990

DIEUDONNÉ-SCHWARTZ THEOREM IN INDUCTIVE LIMITS OF METRIZABLE SPACES II

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(Communicated by William J. Davis)

ABSTRACT. The Dieudonné-Schwartz Theorem for bounded sets in strict inductive limits does not hold for general inductive limits $E = \operatorname{ind}\lim E_n$. It does if all the E_n are Fréchet spaces and for any $n \in N$ there is $m(n) \in N$ such that $\overline{E}_n^{E_p} \subset E_{m(n)}$ for all $p \geq m(n)$. A counterexample shows that this condition is not necessary. When E is a strict inductive limit of metrizable spaces E_n, this condition is equivalent to the condition that each bounded set in E is contained and bounded in some (E_n, ξ_n). Also, some interesting results for bounded sets in inductive limits of Fréchet spaces are given.

Let $E_1 \subset E_2 \subset \cdots$ be a sequence of locally convex spaces and $(E, \xi) = \operatorname{ind}\lim(E_n, \xi_n)$ their inductive limit with respect to the continuous identity maps id: $(E_n, \xi_n) \to (E_{n+1}, \xi_{n+1})$. The Dieudonné-Schwartz Theorem [1, Chapter 2, §12] states that a set $B \subset E$ is ξ-bounded if and only if it is contained and bounded in some (E_n, ξ_n), provided that $(E, \xi) = \operatorname{ind}\lim(E_n, \xi_n)$ is a strict inductive limit and each E_n is closed in (E_{n+1}, ξ_{n+1}). In [2]–[5], it has been extended to inductive limits. For brevity, denote by

(DS) each set B bounded in (E, ξ) is contained in some E_n;

(DST) each set B bounded in (E, ξ) is contained and bounded in some (E_n, ξ_n).

1. It was proved in [5] that if all (E_n, ξ_n) are Fréchet spaces and for any $n \in N$ there is $m(n) \in N$ such that $\overline{E}_n^{E} \subset E_{m(n)}$ then (DST) holds, where $\overline{E}_n^{E}$ is the closure of E_n in (E, ξ); see [5, Theorem 1]. Since the closure of E_n in (E, ξ) may be difficult to construct, the assumption $\overline{E}_n^{E} \subset E_{m(n)}$ is not practical. This situation is remedied in the following.

Theorem 1. *Let all* (E_n, ξ_n) *be Fréchet spaces. Then* (DST) *holds provided that for any* $n \in N$ *there is* $m(n) \in N$ *such that* $\overline{E}_n^{E_p} \subset E_{m(n)}$ *for all* $p \geq m(n)$, *where* $\overline{E}_n^{E_p}$ *is the closure of* E_n *in* (E_p, ξ_p).

Received by the editors June 15, 1988 and, in revised form, April 30, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46A05.
Key words and phrases. Locally convex spaces, (strict) inductive limit, bounded set.

Proof. Assume that $\overline{E}_n^{E_p} \subset E_{m(n)}$ for any $p \geq m(n)$. Then $\overline{E}_n^{E_p}$ is a closed subspace of the Fréchet space (E_p, ξ_p), so $(\overline{E}_n^{E_p}, \xi_p|\overline{E}_n^{E_p})$ is a Fréchet space. Since $\overline{E}_n^{E_p} \subset E_{m(n)}$ and the topology $\xi_p|E_{m(n)}$ is weaker than $\xi_{m(n)}$, $\overline{E}_n^{E_p}$ is a closed subspace of the Fréchet space $(E_{m(n)}, \xi_{m(n)})$. Hence $(\overline{E}_n^{E_p}, \xi_{m(n)}|\overline{E}_n^{E_p})$ is a Fréchet space too. By the open mapping theorem, $(\overline{E}_n^{E_p}, \xi_{m(n)}|\overline{E}_n^{E_p}) = (\overline{E}_n^{E_p}, \xi_p|\overline{E}_n^{E_p})$. Remark that $\overline{E}_n^{E_{m(n)}}$ is closed in $(\overline{E}_n^{E_p}, \xi_{m(n)}|\overline{E}_n^{E_p})$ and dense in $(\overline{E}_n^{E_p}, \xi_p|\overline{E}_n^{E_p}) = (\overline{E}_n^{E_p}, \xi_{m(n)}|\overline{E}_n^{E_p})$, we have $\overline{E}_n^{E_{m(n)}} = \overline{E}_n^{E_p}$. Thus $\overline{E}_n^{E_{m(n)}}$ is ξ_p-closed for any $p \geq m(n)$. Let $(F_n, \eta_n) = (\overline{E}_n^{E_{m(n)}}, \xi_{m(n)}|\overline{E}_n^{E_{m(n)}})$, then (F_n, η_n) is a Fréchet space and F_n is ξ_p-closed for any $p \geq m(n)$. Particularly, F_n is $\xi_{m(n+1)}$-closed, and F_n is closed in $(\overline{E}_{n+1}^{E_{m(n+1)}}, \xi_{m(n+1)}|\overline{E}_{n+1}^{E_{m(n+1)}}) = (F_{n+1}, \eta_{n+1})$. Let $(F, \eta) = \operatorname{ind}\lim(F_n, \eta_n)$. Then all (F_n, η_n) are Fréchet spaces and each F_n is closed in (F_{n+1}, η_{n+1}). It follows from the Dieudonné–Schwartz Theorem that (DST) holds for $(F, \eta) = \operatorname{ind}\lim(F_n, \eta_n)$. Evidently, $(E, \xi) = (F, \eta)$; [see 6, Chapter V, Supplement (3)]. Thus a set $B \subset E$ is ξ-bounded if and only if it is η-bounded if and only if it is contained and bounded in some (F_n, η_n), namely, B is contained and bounded in some (E_m, ξ_m). Hence (DST) holds for $(E, \xi) = \operatorname{ind}\lim(E_n, \xi_n)$.

Corollary 1. *Let all (E_n, ξ_n) be Fréchet spaces. Then* (DST) *holds provided that for any $n \in N$ there is $m(n) \in N$ such that for each $p \geq m(n)$, there is a neighborhood U_p of* 0 *in (E_p, ξ_p) satisfying $(\overline{U_p \cap E_n})^{E_p} \subset E_{m(n)}$.*

Proof. Without loss of generality, we may assume that U_p is an open absolutely convex neighborhood of 0 in (E_p, ξ_p). Let $x \in U_p \cap \overline{E}_n^{E_p}$, then there is a sequence $\{x_i\} \subset E_n$ such that $x_i \xrightarrow{i} x$ in (E_p, ξ_p). Since U_p is ξ_p-open and $x \in U_p$, there is an index i_0 such that $x_i \in U_p$ for any $i \geq i_0$. Thus $x_i \in U_p \cap E_n$ for any $i \geq i_0$ and $x_i \xrightarrow{i} x$ in (E_p, ξ_p), so $x \in (\overline{U_p \cap E_n})^{E_p}$. Namely $U_p \cap \overline{E}_n^{E_p} \subset (\overline{U_p \cap E_n})^{E_p}$. By the hypotheses, $(\overline{U_p \cap E_n})^{E_p} \subset E_{m(n)}$, so $U_p \cap \overline{E}_n^{E_p} \subset E_{m(n)}$. Thus $\overline{E}_n^{E_p} = E_p \cap \overline{E}_n^{E_p} = (\bigcup_{k=1}^{\infty} kU_p) \cap \overline{E}_n^{E_p} = \bigcup_{k=1}^{\infty} k(U_p \cap \overline{E}_n^{E_p}) \subset E_{m(n)}$. By Theorem 1, (DST) holds.

Corollary 2. *Let all (E_n, ξ_n) be Fréchet spaces. Then* (DST) *holds provided that for any $n \in N$ there is $m(n) \in N$ such that $\ulcorner E \urcorner_n^E \subset E_{m(n)}$, where $\ulcorner E \urcorner_n^E$ is the quasi-closure* [7, p. 296] *of E_n in (E, ξ).*

Counterexample. Put $W(x) = \sqrt{1 + x^2}$, $x \in (-\infty, +\infty)$, and $E_n = \{f \in L^2(R): \|f\|_n^2 = \int_R |W^{-n} f|^2 dx < +\infty)$. The norm $\|\cdot\|_n$ makes E_n into a Hilbert space. Let $\mathscr{D}[-n, n] = \{f \in C^{\infty}(R): \operatorname{supp} f \subset [-n, n]\}$ and $\mathscr{D} = \bigcup_{n=1}^{\infty} \mathscr{D}[-n, n]$, then $\mathscr{D}$ is dense in each E_n, so we have $E_{n+p} = \overline{\mathscr{D}}^{E_{n+p}} \subset \overline{E}_n^{E_{n+p}}$

and the hypothesis in Theorem 1 does not hold. But, by Theorem 4 in [3], (DST) holds.

The counterexample shows that the hypothesis in Theorem 1 is not a necessary condition for (DST). However, for strict inductive limits of metrizable spaces, we have the following theorem.

Theorem 2. *Let* $(E,\xi) = \operatorname{ind}\lim(E_n,\xi_n)$ *be a strict inductive limit of metrizable spaces. Then* (DST) *holds if and only if for any* $n \in N$ *there is* $m(n) \in N$ *such that* $\overline{E}_n^{E_p} \subset E_{m(n)}$ *for all* $p \geq m(n)$.

Proof. By Theorem 2 in [5], if (DST) holds then for any $n \in N$ there is $m(n) \in N$ such that $\overline{E}_n^{E} \subset E_{m(n)}$. Since $\overline{E}_n^{E_p} \subset \overline{E}_n^{E}$ for all $p \geq m(n)$, the necessity is proved.

Conversely, assume that the condition is satisfied and there is a set B bounded in (E,ξ) which is not contained in any E_n. Choose a sequence $1 = n_1 \leq m(n_1) < n_2 \leq m(n_2) < n_3 \leq m(n_3) < n_4 \leq \cdots$ such that $\overline{E}_{n_k}^{E_p} \subset E_{m(n_k)} \subset E_{n_{k+1}}$ and $b_k \in B \cap (E_{n_{k+1}} \setminus \overline{E}_{n_k}^{E_p})$ for all $p \geq m(n_k)$, $k \in N$. Since $(1/k)b_k \in E_{n_{k+1}} \setminus \overline{E}_{n_k}^{E_{n_{k+1}}}$ and (E_{n_k},ξ_{n_k}) is a topological subspace of $(E_{n_{k+1}},\xi_{n_{k+1}})$, there is an absolutely convex neighborhood U_{n_k} of 0 in (E_{n_k},ξ_{n_k}) such that $U_{n_{k+1}} \cap E_{n_k} = U_{n_k}$ and $(1/k)b_k \notin U_{n_{k+1}}$; see [8, Lemma 13-3-2]. Put $U = \bigcup_{k=1}^{\infty} U_{n_k}$, then U is a neighborhood of 0 in (E,ξ) which does not contain any $(1/k)b_k$, a contradiction. Thus B is contained in some E_n and (DS) holds. Since $\xi|E_n = (E_n,\xi_n)$ for any $n \in N$, we have (DST) holds.

2. Next we shall give some interesting results for bounded sets in inductive limits of Fréchet spaces.

An absolutely convex subset B of a locally convex space is called a barrelled disk (respectively, Banach disk) if sp[B] with the gauge of B is a barrelled space (respectively, Banach space). Obviously each Banach disk is a barrelled disk. But the converse is not true.

Theorem 3. *Let all* (E_n,ξ_n) *be Fréchet spaces and a set* B *contained and closed in some* (E_n,ξ_n). *If* B *is a barrelled disk* (*or Banach disk*) *then* B *is bounded in* (E_n,ξ_n).

Proof. Let $U_n^{(1)} \supset U_n^{(2)} \supset U_n^{(3)} \supset \cdots$ be a base of closed absolutely convex neighborhoods of 0 in (E_n,ξ_n). Put $E_B = \mathrm{sp}[B]$, then $E_B \subset E_n$. The collection of all sets $(B/2^{k-1}) \cap U_n^{(k)}$, $k = 1,2,3,\cdots$, forms a base of neighborhoods of 0 for some locally convex topology on E_B. We denote the topology on E_B by η. Obviously (E_B,η) is metrizable. Let $\{x_i\}$ be a Cauchy sequence in (E_B,η), then $\{x_i\}$ is also a Cauchy sequence in (E_n,ξ_n). Since (E_n,ξ_n) is complete, there is $x_0 \in E_n$ such that $x_i \xrightarrow{i} x_0$ in (E_n,ξ_n). For any fixed $k \in N$, there is $i_k \in N$ such that $x_i - x_j \in (B/2^{k-1}) \cap U_n^{(k)}$ for $i,j \geq i_k$.

Since $(B/2^{k-1} \cap U_n^{(k)})$ is closed in (E_n, ξ_n) and $x_j \xrightarrow{j} x_0$ in (E_n, ξ_n), we have $x_i - x_0 \in (B/2^{k-1}) \cap U_n^{(k)}$ for $i \geq i_k$. Thus $x_i \xrightarrow{i} x_0$ in (E_B, η). Hence (E_B, η) is a Fréchet space. Suppose that p_B is the gauge of B, then (E_B, p_B) is a barrelled space. Evidently the identity mapping id: $(E_B, \eta) \to (E_B, p_B)$ is continuous. By the open mapping theorem, id is open. Hence for each $k \in N$, there is $\lambda_k > 0$ such that $\lambda_k B \subset (B/2^{k-1}) \cap U_n^{(k)}) \subset U_n^{(k)})$. Namely B is absorbed by each $U_n^{(k)}$, so B is bounded in (E_n, ξ_n). This completes the proof.

Let $(E, \xi) = \operatorname{ind}\lim(E_n, \xi_n)$ be an inductive limit of locally convex spaces and a set B contained in some E_n. If B is ξ_n-bounded then B is ξ_m-bounded for any $m > n$. However B ξ_m-bounded (for some $m > n$) does not imply B ξ_n-bounded because the topology ξ_n is stronger than $\xi_m | E_n$.

Using Theorem 3, we obtain the following two elegant results.

Theorem 4. *Let all (E_n, ξ_n) be Fréchet spaces and an absolutely convex set B contained in E_n. If there is $m > n$ such that B is bounded and closed in (E_m, ξ_m), then B is bounded and closed in (E_n, ξ_n).*

Proof. Since B is closed in (E_m, ξ_m) and the topology ξ_n is stronger than $\xi_m | E_n$, B is closed in (E_n, ξ_n). By the hypothesis, there is $m > n$ such that B is bounded and closed in the Fréchet space (E_m, ξ_m), so B is a Banach disk. By Theorem 3, B is bounded in (E_n, ξ_n).

Theorem 5. *Let all (E_n, ξ_n) be Fréchet spaces and* (DST) *hold. If an absolutely convex set $B \subset E_n$ is bounded and closed in (E, ξ), then B is bounded and closed in (E_n, ξ_n).*

Proof. Since (DST) holds, B is bounded in (E_m, ξ_m) for some $m \geq n$. By the hypotheses, B is ξ-closed, hence also ξ_m-closed. By Theorem 4, B is bounded and closed in (E_n, ξ_n).

Acknowledgement

The author wishes to thank the referee for his helpful suggestions.

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