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A NOTE ON FRÉCHET-MONTEL SPACES

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ABSTRACT. Let E be a Fréchet space and let $C^b(E)$ denote the vector space of all bounded continuous functions on E. It is shown that the following statements are equivalent: (i) E is Montel. (ii) Every bounded continuous function from E into c_0 maps every absolutely convex closed bounded subset of E into a relatively compact subset c_0. (iii) Every sequence in $C^b(E)$ that converges to zero in the compact-open topology also converges uniformly to zero on absolutely convex closed bounded subsets of E.

1. INTRODUCTION

It is well known that a Fréchet space E is Montel iff it is separable and every weak*-null sequence in E' converges uniformly to zero on bounded subsets of E. Because of the Josefson–Nissenzweig theorem H. Jarchow asked in [6, p. 247] if this result is true without the separability condition on E. The property that every weak*-null sequence in E' converges uniformly to zero on bounded subsets of E holds iff every continuous linear mapping from E into c_0 maps every bounded subset of E into a relatively compact subset of c_0. In this note we prove that a Fréchet space E is Montel iff every bounded continuous function from E into c_0 maps every absolutely convex closed bounded subset of E into a relatively compact subset of c_0. Using this result, we obtain that a Fréchet space E is Montel iff every sequence in $C^b(E)$ that converges to zero in the compact-open topology also converges uniformly to zero on absolutely convex closed bounded subsets of E.

For basic definitions and notations not given below, see[6]. For a Hausdorff locally convex space (short lcs) E, $\mathscr{U}_E$ denotes a fundamental system of absolutely convex closed neighbourhoods of zero in E. For $U \in \mathscr{U}_E$, we write E_U for the normed space canonically associated with U and φ_U for the corresponding quotient mapping $E \to E_U$. We shall also consider φ_U as a map from E into $\widehat{E}_U$ the completion of E_U. Let $C^b(E)$ be the vector space of all bounded continuous functions on E. By E' we mean the topological dual of E. For E lcs we take the notations E'_{pc} and E'_β, if E' is equipped with the

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topology of uniform convergence on the precompact and on the bounded subsets of E, respectively. If every null sequence in $\sigma(E', E)$ is equicontinuous, then E is called *c_0-barrelled*. Every barrelled space is c_0-barrelled. Further let $\mathscr{B}_E$ denote a fundamental system of absolutely convex closed bounded subsets of E. When $A \in \mathscr{B}_E$ let E_A denote the associated normed space. If A, $B \in \mathscr{B}_E$ and $A \subset B$ then there is a unique continuous linear extension $i_{AB}: \widehat{E}_A \to \widehat{E}_B$ of the canonical inclusion $E_A \to E_B$ to the completions. If E is a quasi-complete space, then E_A is a Banach space for every $A \in \mathscr{B}_E$.

2. A characterization of Fréchet–Montel spaces

Let E be a lcs and let $\mathscr{A}$ be an ideal of Banach space operators. Then E is called an *$\mathscr{A}$-space* [6], if for each $U \in \mathscr{U}_E$ there exists a $V \in \mathscr{U}_E$ with $V \subset U$ such that $\varphi_{UV} \in \mathscr{A}(\widehat{E}_V, \widehat{E}_U)$. A bounded operator from a Banach space into another Banach space is called a *Rosenthal operator*, if the image of the unit ball is conditionally weakly compact. Recall that a subset B of a lcs is called *conditionally weakly compact*, if every sequence in B has a weak Cauchy subsequence. The Rosenthal operators between Banach spaces form a closed, surjective and injective ideal $\mathscr{R}$. In [9] we have called a subset B of a lcs E *limited*, if every equicontinuous, $\sigma(E', E)$-null sequence in E' converges uniformly to zero on B. Thus every bounded subset of E is limited iff every equicontinuous, $\sigma(E', E)$-null sequence in E' converges uniformly to zero on bounded subsets of E. Let E and F be Banach spaces. A bounded operator $T: E \to F$ is called *limited* [1], if T takes the unit ball of E to a limited subset of F. The limited operators between Banach spaces form a closed, surjective ideal $\mathscr{L}im$ that is not injective. The injective hull of $\mathscr{L}im$ is $\mathscr{R}$ [9]. In [8] we have proved that the product of three limited operators is compact. We start with the following important consequences of this result:

Proposition 1. *Let E be a* lcs. *Then*:

(i) *E'_β is a Schwartz space iff for every $A \in \mathscr{B}_E$ there exists a $B \in \mathscr{B}_E$, $A \subset B$, such that $i_{AB}: \widehat{E}_A \to \widehat{E}_B$ is limited.*

(ii) *E is a Schwartz space iff it is quasi-normable and every bounded subset of E is limited.*

Proof. (i) By [10] E'_β is a Schwartz space iff for every $A \in \mathscr{B}_E$ there exists a $B \in \mathscr{B}_E$, $A \subset B$, such that $i_{AB}: \widehat{E}_A \to \widehat{E}_B$ is compact. Thus the statement is a consequence of the fact that the product of three limited operators is compact. (ii) This is Corollary 3 in [8] which says that E is Schwartz iff it is quasi-normable and every equicontinuous, weak*-null sequence in E' is also E'_β-null convergent.

For our next proposition we need some lemmata.

Lemma 2. *Let E be a reflexive space such that E'_{pc} is c_0-barrelled. Then every bounded subset of E'_β is limited.*

Proof. Let $A \subset E'_\beta$ be bounded. Then A is equicontinuous, since E is quasi-barrelled. By Alaoglu–Bourbaki's theorem A is relatively compact in E'_{pc}, and consequently limited in E'_{pc} [9]. Now let (u_n) be an equicontinuous, weak*-null sequence in $(E'_\beta)'$. Since $E = (E'_\beta)'$ implies that $E = (E'_{pc})'$ we get that (u_n) is an equicontinuous, weak*-null sequence in $(E'_{pc})'$. Hence (u_n) converges uniformly to zero on A, i.e. A is limited in E'_β.

Lemma 3. *Let E be a quasi-complete space. Then E has the Schur property, i.e. every weak-null sequence (x_n) in E converges to zero in E, iff E'_{pc} is c_0-barrelled.*

Proof. Since E is quasi-complete, $E = (E'_{pc})'$. Suppose that E'_{pc} is c_0-barrelled. Let $x_n \to 0$ in $\sigma(E, E') = \sigma((E'_{pc})', E'_{pc})$. Hence $x_n \in K^{\circ\circ}$, where K is a relatively compact subset of E. This implies that $x_n \to 0$ in E, since the topology of E and $\sigma(E, E')$ coincide on the compact subset $K^{\circ\circ}$ of E. Conversely, suppose that E has the Schur property. Let $u_n \to 0$ in $\sigma((E'_{pc})', E'_{pc}) = \sigma(E, E')$. Then $K := \{u_n : n \in N\}$ is a relatively compact subset of E and hence equicontinuous.

Proposition 4. *Let E be a quasi-complete, barrelled space such that E'_β is quasi-normable. The following statements are equivalent*:

(i) *Every bounded subset of E'_β is limited.*

(ii) *E'_β is Schwartz.*

(iii) *E is Montel.*

(iv) *E is reflexive and E'_{pc} is c_0-barrelled.*

Proof. (i) $\Leftrightarrow$ (ii) by Proposition 1 (ii). Since every bounded subset of E is precompact, when E'_β is Schwartz [10], (ii) $\Rightarrow$ (iii). (iii) $\Rightarrow$ (iv) by Lemma 3. (iv) $\Rightarrow$ (i) by Lemma 2.

This proposition is not valid if we drop the assumption that E is quasi-barrelled. This follows from the following example constructed by A. Garcia and J. Gómez in [5]. Let F be the Banach space l_∞, (x_n) a weak null sequence in F, such that $\|x_n\| = 1$ for each n and $\xi = \{S \subset F : S$ is finite or $S = \{x_n : n \in N\}\}$. Let E be the space F' endowed with the locally convex topology, having as a subbase at zero the sets S^0, when S ranges over ξ. Then $E'_\beta = l_\infty$, E is semi-reflexive with the Schur property but not semi-Montel. The Schur property of E follows from the fact that l_∞ is a Grothendieck space with the Dunford–Pettis property.

If the equicontinuity is removed from the definition of a limited set, then relatively compactness does not imply limitedness. This follows from the proof of Lemma 2, since otherwise we would obtain that e.g. all Hilbert spaces are finite-dimensional. In the class of c_0-barrelled spaces these two concepts of limited sets of course coincide. If E is a Fréchet space, then E'_β is a DF-space which in turn is a quasi-normable space. For a Fréchet space E we obtain by the Eberlein–Smulian theorem [6, 9.8.3] that E has the Schur property iff every relatively weakly compact subset of E is relatively compact.

Proposition 5. *Let E be a quasi-complete space. Then E is semireflexive if it is a $\mathscr{R}$-space and every bounded subset of E is limited.*

Proof. Assume that E is a $\mathscr{R}$-space and every bounded subset B of E is limited. Since every Rosenthal operator factors through a Banach space not containing l_1 [2] it follows immediately that E is a $\mathscr{R}$-space iff it has a zero-basis $\mathscr{U}_E$ such that all the Banach spaces $\widehat{E}_U$, $U \in \mathscr{U}_E$, do not contain a copy of l_1. Hence, since $\varphi_U(B)$ is limited in $\widehat{E}_U$ for every $U \in \mathscr{U}_E$, it follows from Proposition 7 in [1] that $\varphi_U(B)$ is relatively weakly compact in $\widehat{E}_U$ for every $U \in \mathscr{U}_E$. Since E is quasi-complete, this means that E is semi-reflexive by Proposition 7.5.1 in [7].

Next we want to point out that there exists a Frécet–Montel space which is not a $\mathscr{R}$-space. This means that the property that every bounded subset of a Fréchet space E is limited does not imply that E is a $\mathscr{R}$-space. Indeed, let E denote Köthe's example of a Fréchet–Montel space have l_1 as a quotient space [6, 11.6.4]. Every quotient of a $\mathscr{R}$-space is a $\mathscr{R}$-space by Proposition 21.1.5 in [6]. Since a Banach space is a $\mathscr{R}$-space iff it does not contain a copy of l_1, we get that E is not a $\mathscr{R}$-space. But, *if E is a Fréchet space such that every bounded subset in E is limited, then for every $A \in \mathscr{B}_E$ there exists a $B \in \mathscr{B}_E$, $A \subset B$, such that $i_{AB}\colon E_A \to E_B$ is Rosenthal.* In fact, every limited subset of a Fréchet space is conditionally weakly compact [9]. Hence every bounded subset of E is conditionally weakly compact. Now the statement follows from Theorem 7.3.3 in [7].

In order to prove our main result we need the following extension of Tietze's theorem due to J. Dugundji [3]: *Let E be an arbitrary metric space, A a closed subset of E, G a* lcs *and $f\colon A \to G$ a continuous function. Then there exists a continuous extension $F\colon E \to G$ of f such that $F(E)$ is contained in the convex hull of $f(A)$.*

Proposition 6. *Let E be a Fréchet space. Then E is Montel iff every bounded continuous function from E into c_0 maps every absolutely convex closed bounded subset of E into a relatively compact subset of c_0.*

Proof. If E is Montel, then every closed bounded subset of E is compact. Since the continuous image of a compact set is compact, the assertion is proved. Conversely, suppose that the condition is fulfilled. Since a Fréchet space E is Montel iff E'_β is Schwartz (Proposition 4), we get by Proposition 1 (i) that E is Montel, if every $A \in \mathscr{B}_E$ is limited in the Banach space E_B for some $B \in \mathscr{B}_E$ with $A \subset B$. In [1] J. Bourgain and J. Diestel have noticed that A is limited in E_B iff for every $T \in L(E_B, c_0)$ we have that $T(A)$ is relatively compact in c_0. Now let $A \in \mathscr{B}_E$. Since E is Fréchet there is a countable zero-basis of closed, absolutely convex zero-neighborhoods (U_n) in E, and for every n there exists a $\alpha_n > 0$ such that $A \subset \alpha_n U_n$. We now choose $\beta_n \geq \alpha_n$ such that the sequence (α_n/β_n) tends to zero and put $B = \bigcap_n \beta_n U_n$. Then $B \in \mathscr{B}_E$ and $A \subset B$. Given $\varepsilon > 0$ there is a positive integer j such that $\alpha_n \leq \varepsilon\beta_n$ for $n \geq j$. Hence $A \subset \varepsilon\beta_n U_n$ for $n \geq j$. Next, we have that $\bigcap_{n=1}^{j-1} \varepsilon\beta_n U_n$ is a zero-neighborhood in E and consequently there is a positive integer m with $U_m \subset \bigcap_{n=1}^{j-1} \varepsilon\beta_n U_n$. Then $A \cap U_m \subset \varepsilon\beta_n U_n$ for all n, i.e. $A \cap U_m \subset \varepsilon B$. This means that the Banach space E_B induces on A the same topology as E. Let now $T \in L(E_B, c_0)$ be arbitrary. The restriction map $T|A\colon A \to c_0$ is continuous, when A is endowed with the topology induced by E. Since A is bounded in E_B, $T(A)$ is also bounded in c_0. Now we get by the above extension theorem that there exists a continuous function $F\colon E \to c_0$ which maps E into a bounded subset c_0. Hence $T(A) = F(A)$ is relatively compact in c_0 by the assumption. Thus we have proved that E is Montel.

In the next corollary we give a positive answer to the analogue of Jarchow's question for bounded continuous functions. We use the well-known fact that a bounded subset A of c_0 is relatively compact iff $\sup_{x\in A} |e'_n(x)| \to 0$, when $n \to \infty$, where e'_n is the standard basis in l_1.

Corollary 7. *Let E be a Fréchet space. Then E is Montel iff every sequence in $C^b(E)$ that converges to zero in the compact-open topology also converges uniformly to zero on absolutely convex closed bounded subsets of E.*

Proof. Let $B \in \mathscr{B}_E$ and let $f\colon E \to c_0$ be a bounded continuous function. Put $g_n = e'_n \circ f$, where e'_n is the standard basis in l_1. Then $g_n \in C^b(E)$ and $g_n \to 0$ in $C^b(E)_{co}$, since if $K \subset E$ is compact, then $f(K)$ is compact in c_0 and consequently $\sup_{x\in K} |g_n(x)| \to 0$, when $n \to \infty$. By the assumption we have that $\sup_{x\in B} |e'_n(f(x))| \to 0$, when $n \to \infty$. This means that $f(B)$ is relatively compact in c_0. The converse implication is obvious.

Let E be a Fréchet space and suppose that every $C^b(E)_{co}$-null sequence in $C^b(E)$ converges uniformly to zero on absolutely convex closed bounded subsets of E. Then E is a Fréchet–Montel space, and consequently separable. Hence $C(E)_{co}$ is ultrabornological and separable, since E is realcompact. Notice also that a metrizable space of nonmeasurable cardinal is realcompact and that the subspace $C^b(R)_{co}$ of $C(R)_{co}$ is not barrelled, since $\{f \in C^b(R)\colon |f(x)| \leq 1,\ x \in R\}$ is a barrel but not a zero-neighborhood.

We conclude this note by a result concerning the ideal of Grothendieck operators. Let E and F be Banach spaces. An operator $T\colon E \to F$ is called a *Grothendieck operator*, if every weak*-null sequence (y'_n) in F' is mapped by T' into a weak-null sequence $(T'(y'_n))$ in E'. If T is the identity, then E is a Grothendieck space. The Grothendieck operators between Banach spaces form a surjective ideal $\mathscr{G}$. Notice that every Grothendieck operator is limited, if E' has the Schur property.

The following result can be proved in the same way as Theorem 2.1 in [4].

Proposition 8. *Let E and F be Banach spaces and G a Banach space containing a copy of c_0. If $T \otimes id_G: E\hat{\otimes}_\varepsilon G \to F\hat{\otimes}_\varepsilon G$ is a Grothendieck operator, then $T: E \to F$ is limited.*

Since every $\mathscr{L}im$-space is a Schwartz space by Corollary 2 in [8] we get by the well-known representation of ε-tensor products as projective limits [6,16.3.3] the following:

Corollary 9. *Let E be a* lcs *and G a Banach space containing a copy of c_0. If $E\hat{\otimes}_\varepsilon G$ is a $\mathscr{G}$-space, then E is a Schwartz space.*

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