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ON A PROBLEM OF G. G. LORENTZ REGARDING THE NORMS OF FOURIER PROJECTIONS

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ABSTRACT. For any $0 < \alpha < \frac{1}{2}$ we construct a sequence of integers $(\mu_1, \dots, \mu_n, \dots)$ such that the norms of Fourier projections

$$F_N = \sum_1^N e^{i\mu_j\theta} \otimes e^{i\mu_j t}: C_{[-\pi,\pi]} \to C_{[-\pi,\pi]}$$

grow as N^α. This answers a question of Prof. G. G. Lorentz.

I. INTRODUCTION

Let $M = (\mu_1, \dots, \mu_n) \subset \mathbf{N}$ be an increasing sequence of integers. Let $C_{[-\pi,\pi]}$ be the space of continuous functions f on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$. For every $f \in C_{[-\pi,\pi]}$ let $F_M f$ be the Fourier projection of f:

$$F_M f = \frac{1}{2\pi} \sum_{j=1}^n \left(\int_{-\pi}^{\pi} f(\theta) e^{i\mu_j\theta}\, d\theta \right) e^{i\mu_j\theta}.$$

The celebrated solution of the Littlewood conjecture (cf. [2]) implies (cf. [3])

$$(1) \qquad C_1 \log(\#M) \le \|F_M\| \le C_2\sqrt{\#M}.$$

We use $\#M$ to denote the cardinality of M. Here and throughout this paper we use the letter C with subscripts to denote positive constants that do not depend on M.

It is also well known (cf. [1], [3]) that the lower and upper bounds in (1) are attained with M being an arithmetic and lacunary sequence, respectively:

Proposition 1. *There exist constants C_3, C_4, C_5, C_6 such that*

(a) *if M is an arithmetic sequence then* $C_3 \log(\#M) \le \|F_M\| \le C_4 \log(\#M)$

(b) *if $M = (\lambda_1, \dots, \lambda_n)$ is such that $(\lambda_k + 1)/\lambda_k > 2$ then* $C_5\sqrt{\#M} \le \|F_M\| \le C_6\sqrt{\#M}$.

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In the 1986 Edmonton conference on constructive approximation, Professor G. G. Lorentz posed the following:

Problem. Let $0 < \alpha < \frac{1}{2}$ be a real number. Exhibit a sequence of integers $M = (\mu_1, \dots, \mu_n, \dots)$ so that

$$\|F_{M_n}\| \asymp n^\alpha \text{ as } n \to \infty,$$

where $M_n = (\mu_1, \dots, \mu_n)$.

In this note we solve the problem.

The idea of the construction is to combine the lacunary sequences and arithmetic progressions in such a way that the cardinality of the set of arithmetic sequences is sufficiently large to be a major factor in $\#M_n$, while the cardinality of the lacunary sequence determines the norms of $\{F_{M_n}\}$, thus achieving an arbitrary balance between the two.

For a positive number a we use $\lfloor a \rfloor$ to denote the integer part of a. We will need the following simple estimates:

Proposition 2. *Let* $a > 1$. *There exist constants* $C_7(a)$, $C_8(a)$, $C_9(a)$, $C_{10}(a)$ *such that*

$$C_7(a)n^{a+1} \le \sum_{k=1}^{n} \lfloor k^a \rfloor \le C_8(a)n^{a+1} \tag{2}$$

$$C_9(a)n \log n \le \sum_{k=1}^{n} \log\lfloor k^a \rfloor \le C_{10}(a)n \log n. \tag{3}$$

Proof. To prove (2) it suffices to observe

$$\sum_1^n \lfloor k^a \rfloor \le \sum_1^n \lfloor n^a \rfloor = n\lfloor n^a \rfloor \le n \cdot n^a.$$

On the other hand

$$\sum_1^n \lfloor k^a \rfloor \ge \sum_{\lfloor n/2 \rfloor}^{n} \lfloor k^a \rfloor \ge \sum_{\lfloor n/2 \rfloor}^{n} \left\lfloor \left(\frac{n^a}{2}\right) \right\rfloor \ge \left(\frac{1}{2}\right)^a \cdot \frac{1}{2} n \cdot n^a - 1.$$

The proof of (3) is identical:

$$\sum_1^n \log\lfloor k^a \rfloor \le \sum_1^n \log = \lfloor n^a \rfloor \le n \log(n^a) = an \log n.$$

$$\sum_1^n \log\lfloor k^a \rfloor \ge \sum_{\lfloor n/2 \rfloor}^{n} \log\left(\frac{n}{2}\right)^2 \ge C_9(a)n \log n. \square$$

II. Construction

Given an $\alpha : 0 < \alpha < \frac{1}{2}$ let

$$a = 3/2\alpha - 1 : a > 2. \tag{4}$$

We start with a lacunary sequence of integers

$$\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n, \dots)$$

that satisfies

$$\lambda_{m+1} - \lambda_m > \lfloor n^a \rfloor \qquad \text{for all } m \geq n. \tag{5}$$

(For instance $\lambda_m = m^m$ will do).

We now construct the set M in an inductive manner.

Choose $\mu_1 = \lambda_1$. On the nth step we introduce the first n^2 available integers out of the set Λ (say $\lambda_{m+1}, \dots, \lambda_{m+n^2}$) and include in addition

$$\lambda_{m+n^2} + 1, \quad \lambda_{m+n^2} + 2, \dots, \quad \lambda_{m+n^2} + \lfloor n^a \rfloor.$$

By (5) we have

$$\lambda_{m+n^2} + \lfloor n^a \rfloor < \lambda_{m+n^2+1}.$$

Now our set consists of a lacunary set

$$\Lambda_n = (\lambda_1, \dots, \lambda_{m+n^2}) \tag{6}$$

of length

$$\#\Lambda_n = \sum_{k=1}^{n} k^2 \tag{7}$$

and of a set of arithmetic sequences $\mathscr{A}_1, \dots, \mathscr{A}_n$ of length

$$\#\mathscr{A}_j = \lfloor j^a \rfloor.$$

Theorem. *For the set M constructed above let $F_N := F_{M_N}$. There exist constants C, $C_0 > 0$ such that*

$$C_0 N^\alpha \leq \|F_N\| \leq C N^\alpha.$$

Proof. Let N be an integer and let us assume that the number μ_N had been added to the set M during the $(N+1)$st step of the construction of the set M. Then

$$\#\Lambda_n + \sum_{k=1}^{n} (\#\mathscr{A}_k) \leq N \leq \#\Lambda_{n+1} + \sum_{k=1}^{n+1} (\#\mathscr{A}_k). \tag{8}$$

Using (6), (7), and (2) we have

$$\begin{aligned} N &\geq \sum_{k=1}^{n} k^2 + \sum_{j=1}^{n} \lfloor j^a \rfloor \\ &\geq C_7(2) n^3 + C_7(a) n^{a+1} \\ &\geq C_{11} n^{a+1} \qquad (\text{since } a > 2). \end{aligned}$$

On the other hand

$$\begin{aligned} N &\leq C_8(2)(n+1)^3 + \sum_{j=1}^{n+1} \lfloor j^a \rfloor \\ &\leq C_8(2)(n+1)^3 + C_8(a)(n+1)^{a+1} \\ &\leq C_{12} n^{a+1}. \end{aligned}$$

Hence we have

$$C_{11}n^{a+1} \le N \le C_{12}n^{a+1}. \tag{9}$$

Next we estimate the norms $\|F_N\|$.

Notice that $M_N = \Lambda_n \cup (\bigcup_{k=1}^n \mathscr{A}_k) \cup \Lambda^p \cup \mathscr{A}^q$ where Λ^p is a lacunary sequence of length $p \le (n+1)^2$ and $\mathscr{A}^q$ is an arithmetic sequence of length $q \le \lfloor (n+1)^a \rfloor$. Hence

$$\|F_N\| = \|F_{\Lambda_n \cup \Lambda^p} + \sum_{k=1}^n F_{\mathscr{A}_k} + F_{\mathscr{A}} q\|,$$

Since $\Lambda_n \cup \Lambda^p$ (is a lacunary set) we have (by Proposition 1(a))

$$\begin{aligned}
\|F_N\| &\ge \|F_{\Lambda_n \cup \Lambda^p}\| - \left(\sum_{k=1}^n \|F_{\mathscr{A}_k}\|\right) - \|F_{\mathscr{A}} q\| \\
&\ge C_5 \left[\#(\Lambda_n \cup \Lambda^p)\right]^{1/2} - C_4 \sum_{k=1}^n \log(\#\mathscr{A}_k) - C_4 \ \log(\#\mathscr{A}_q) \\
&\ge C_5 \left(\left(\sum_{k=1}^n k^2\right) + p\right)^{1/2} - C_4 \sum_{k=1}^n \log\lfloor k^a \rfloor - C_4 \log\lfloor q^a \rfloor \\
&\ge \left[C_5 C_7(2) n^3\right]^{1/2} - C_4 C_{10}(a) n \ \log\ n - C_4 \cdot a^2 \ \log(n+1) \\
&\ge C_{14} n^{3/2}.
\end{aligned}$$

Similarly

$$\|F_N\| \le \|F_{\Lambda_n \cup \Lambda^p}\| + \sum_{k=1}^n \|F_{\mathscr{A}_k}\| + \|F_{\mathscr{A}} q\| \le C_{13} n^{3/2}.$$

Hence

$$C_{14} n^{3/2} \le \|F_N\| \le C_{13} n^{3/2}. \tag{10}$$

Combining (9) and (10) we have

$$\|F_N\| \le \left(C_{13} n^{3/2}\right) \le C_{13} \left(\frac{N}{C_{11}}\right)^{3/2(a+1)} \le C N^{3/2(a+1)} = C N^\alpha$$

as well as

$$\|F_N\| \ge C_{14} n^{3/2} \ge C_{14} \left(\frac{n}{C_{12}}\right)^{3/2(a+1)} \ge C_0 N^\alpha. \quad \square$$

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