STABLE RANK OF SUBALGEBRAS OF THE DISC ALGEBRA

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Abstract. We calculate the stable rank of subalgebras (in the algebraic sense) of the disc algebra. We extend the results to subalgebras of \( A(\bar{G}) \), where \( G \) is a bounded domain whose boundary consists of finitely many closed nonintersecting Jordan curves.

Introduction

The concept of the stable rank of a ring, introduced, by H. Bass [1], has been very useful in treating some problems in algebraic \( K \)-theory. In a serial of papers G. Corach and F. D. Suárez calculated the stable rank of Banach algebras of holomorphic functions. Among them are the well-known algebras \( A(K) \) of all continuous complex valued functions on a compact set \( K \subset \mathbb{C} \) which are analytic in its interior. However, the stable rank of the disc algebra \( A(\mathbb{D}) \) was first calculated by P. Jones, D. Marshall and T. Wolff [8] by an entirely different technique.

In this paper we study the stable rank of subalgebras of the disc algebra \( A(\mathbb{D}) \). Generalizations to subalgebras of \( A(\bar{G}) \) are given, where \( G \) denotes a bounded domain whose boundary consists of finitely many nonintersecting closed Jordan curves. However, the best results are obtained for subalgebras of the disc algebra \( A(\mathbb{D}) \).

Since our algebras need not be normable, our results are generalizations of results of G. Corach and F. D. Suárez, see [3]. Also this paper complements the results in [10].

1. It is well known that the group of units in Banach algebras is open. Unfortunately, this feature is lost in the general case of a topological algebra. This led to the following definitions:

A topological algebra \( A \) is called a \( Q \)-algebra if the set of units, \( A^x \), is open in \( A \).

In this paper we consider complex, commutative \( Q \)-algebras with unit element being denoted by 1.
Given a $Q$-algebra $A$, an element $a \in A^n$ is called unimodular if there exists $b \in A^n$ such that

$$\langle b, a \rangle := \sum_{i=1}^{n} b_i a_i = 1.$$ 

We denote by $U_n(A)$ the set of unimodular elements of $A^n$. Finally, $a = (a_1, \ldots, a_n) \in U_n(A)$ is called reducible if there exists $x_1, \ldots, x_{n-1}$ in $A$ such that

$$(a_1 + x_1 a_n, \ldots, a_{n-1} + x_{n-1} a_n) \in U_{n-1}(A).$$

The stable rank of $A$, denoted by $sr(A)$, is the least integer $n$ such that every $a \in U_{n+1}(A)$ is reducible.

From the theory of the stable rank of $Q$-algebras we mention the following fact; see [9, p. 18, Korollar 1] or [4, Proposition 1]. (The proof given there is also valid in $Q$-algebras.)

**Proposition 1.1.** Suppose that $A$ is a $Q$-algebra, $\gamma: [0,1] \to C$ is a continuous curve and let $\Gamma: [0,1] \to U_2(A)$, $\Gamma(t) := (a - \gamma(t), b)$ such that $(a - \gamma(0), b)$ is reducible. Then $(a - \gamma(1), b)$ is reducible, too.

2. Let $K$ be a compact set of the plane $C$ and let $A(K)$ denote the Banach algebra of all complex valued continuous functions which are holomorphic in the interior $K^0$ of $K$.

Let $A$ be a subalgebra of $A(K)$ in the algebraic sense. We say that the Nullstellensatz holds in $A$ iff the following condition holds for every $n$: $(f_1, \ldots, f_n)$ is unimodular iff the functions $f_1, \ldots, f_n$ have no common zero in $K$. If the last condition is known to hold for $n = 1, 2$, we say that the weak Nullstellensatz holds in $A$. If, finally, this condition is known to be true for $n = 1$ then $A$ is said to be inversionally closed. For example, the Nullstellensatz holds in the well-known disc algebra $A(D)$.

Suppose now that the weak Nullstellensatz holds in the algebra $A \subset A(K)$. The inversion set $I(f, g)$ of $f, g \in A$ is defined to be the set of all complex numbers $a$ such that the vector $(f - a, g)$ is unimodular, for short,

$$I(f, g) = \{a \in C: (f - a, g) \in U_2(A)\}.$$ 

Since the weak Nullstellensatz holds in $A$, we can restate this as follows:

$$C \setminus I(f, g) = f(Z_g),$$

where $Z_g$ denotes the set of zeros of $g$ in $K$. This identity shows that every number $\lambda \in C \setminus f(K)$ is a member of the inversion set of $I(f, g)$, moreover, $(f - \lambda, g)$ is reducible, since the algebra $A$ is inversionally closed.

**Remarks.** If the weak Nullstellensatz holds in $A \subset A(K)$, then the algebra $A$, endowed with the topology of uniform convergence, is a $Q$-algebra.

The author has no example of an algebra $A$ for which the weak Nullstellensatz holds, but the Nullstellensatz does not hold.

In view of Proposition 1.1 the inversion set $I(f, g)$ plays an important role.
In this paragraph, we will consider mainly subalgebras of the algebra $H^{1,1}(\mathbb{D})$, where $H^{1,1}(\mathbb{D})$ denotes all those functions of the disc algebra $A(\mathbb{D})$ whose boundary function is absolutely continuous, i.e.

$$H^{1,1}(\mathbb{D}) := \{f \in A(\mathbb{D}): f|_{\partial \mathbb{D}} \text{ is absolutely continuous}\}.$$ 

Further information on $H^{1,1}(\mathbb{D})$ is available in [5], Theorem 3.10.

**Theorem 2.1.** Let $A$ be a subalgebra of $H^{1,1}(\mathbb{D})$ in which the weak Nullstellensatz holds, and suppose that $(f,g)$ is unimodular, where $g$ is not the zero function. Then the complement $C \setminus I(f,g)$ of the inversion set is totally disconnected.

**Proof.** Suppose on the contrary that there exists a closed connected subset $J$ of the compact set $C \setminus I(f,g)$ which contains more than one point. Since the weak Nullstellensatz holds in $A$, we have

$$C \setminus I(f,g) = f(Z_g).$$

Moreover, since $J$ does not have isolated points, for every given point $w \in J$ there exists a sequence $(z_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} f(z_n) = w, \quad g(z_n) = 0 \quad (n \in \mathbb{N}).$$

The identity theorem for analytic functions implies that every accumulation point of $(z_n)_{n \in \mathbb{N}}$ has unit modulus. So there exists a compact set $E \subset \partial \mathbb{D}$ such that

$$(1) \quad J = f(E), \quad g(E) = \{0\}.$$ 

As a subset of a set of Lebesgue measure zero, $E$ also has Lebesgue measure zero.

Let $p_x$ resp. $p_y$ denote the projection on the real resp. imaginary axis.

Since both projections are continuous and $J$ is connected, the sets $p_x(J)$ and $p_y(J)$ are intervals on the real resp. imaginary axis. On the other hand, they are the absolutely continuous image of the set $E \subset \partial D$ which has Lebesgue measure zero. So they also have Lebesgue measure zero and consist therefore of exactly one point. Thus we have the contradiction that $J$ consists of one point.

This theorem is the main ingredient in proving the following result.

**Theorem 2.2.** Let $A$ be a subalgebra of $H^{1,1}(\mathbb{D})$ in which the weak Nullstellensatz holds. Then its stable rank is one.

**Remarks.** It should be noted that no assumptions on topology for the algebra $A$ are made.

To prove Theorem 2.2 we use the fact that $f(Z_g)$ is totally disconnected. This argument has been used before, implicitly or explicitly, see [2, Theorem 1.13] or [4, Corollary 2].

**Proof.** We have to show that every unimodular vector $(f,g)$ is reducible. Since the weak Nullstellensatz holds in $A$, this is equivalent to the existence of a
function $h \in A$ such that $f + hg$ is zerofree in $\overline{D}$. Of course, we can assume that $g$ is not the zero function. By Proposition 1.1 and the subsequent remark it is enough to exhibit a path $\gamma$ such that $(f - \gamma(t), g)$ is unimodular, $\gamma(0) = 0$ and $(f - \gamma(1), g)$ is reducible. By definition $(f - a, g)$ is unimodular iff $a \in I(f, g)$. Theorem 2.1 implies that $C \setminus I(f, g)$ is totally disconnected and compact, that is, its (covering) dimension is zero ([7], p. 20, Section A). By a result of plane topology, see [7], Theorem IV.4, the open set $I(f, g)$ is connected (and unbounded). Choose any point $b \notin f(\overline{D})$ and a path $\gamma \subset I(f, g)$ joining $0$ and $b$. Since $A$ is inversionally closed we have $b \in I(f, g)$. An inspection shows that $\gamma$ has the required properties. 

This theorem implies that the following algebras have stable rank one:

$$A^N(\overline{D}) := \{f \in A(\overline{D}) : f^{(n)} \in A(\overline{D}), n = 1, \ldots, N\};$$

$$A^\infty(\overline{D}) := \{f \in A(\overline{D}) : f^{(n)} \in A(\overline{D}) \text{ for all numbers } n\}.$$ 

Note that the topological algebra $A^\infty(\overline{D})$ is not normable, [11].

3. It is natural to ask whether Theorem 2.1 holds in general domains.

**Theorem 3.1.** Let $G$ be a bounded domain whose boundary consists of finitely many nonintersecting closed Jordan curves. Assume further that $f$ is holomorphic in $\overline{G}$ and $g \in A(\overline{G})$ is not identical zero. Then the compact set $f(Z_g)$ is totally disconnected.

**Proof.** Suppose on the contrary that there exists a continuum $J$ in $f(Z_g)$. Since $J$ cannot contain isolated points, the identity theorem for holomorphic functions implies the existence of a compact set $K \subset \partial G$ such that $J = f(K)$ and $g(K) = \{0\}$.

It is well known from topology that there exist continua $J' \subset J$ with diameter less than a given positive number $\epsilon$. Since $f'$ is also holomorphic in $\overline{G}$, we can therefore assume that no point of $J$ is a critical point of $f$, that is, we have $J = f(K)$, $g(K) = \{0\}$ and $0 \notin f'(K)$.

Now the function $f$ is locally injective on $K$, and a standard argument shows the existence of compact sets $K_j$, $j = 1, \ldots, n$ such that $K = \bigcup_{j=1}^n K_j$ and $f|_{K_j}$ is injective.

This yields $J = f(K) = \bigcup_{j=1}^n f(K_j)$.

Now the set $Z_g \cap \partial G$ is totally disconnected. (A continuum would be on exactly one Jordan curve $\Gamma$, so it is an arc. The well-known two-constant theorem (see [6], p. 299, Satz 2) implies the contradiction $g = 0$. 

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Every compact subset $K_j$ of $Z_g \cap \partial G$ is also totally disconnected. Their (covering) dimension is therefore zero ([7], p. 20, Section A). Since the functions $f|_{K_j}$ are homeomorphisms, the compact sets $f(K_j)$ also have dimension zero.

By the sum theorem in dimension theory $J$ has dimension zero, too ([7], Theorem III.2). Because $J$ is compact, this implies that $J$ is totally disconnected. But by assumption $J$ was a continuum, which is a contradiction. Therefore $f(Z_g)$ is totally disconnected. \(\Box\)

We will use this result to calculate the stable rank of some algebras.

Let $K$ be a compact plane set. We say that a subalgebra $A$ of $A(K)$ is a holomorphically generated $Q$-algebra, for short, $HGQ$-algebra iff the following two conditions hold:

(i) $A$ is a $Q$-algebra.

(ii) There exists a set $L$ of functions holomorphic on $K$ such that $1 \in L$, and in each neighborhood of a suitable element of $A$ we find a member of $L$, that is the closure of $L$ is $A$.

For example, the disc algebra $A(D)$ is a $HGQ$-algebra.

**Theorem 3.2.** Let $G$ be a bounded domain whose boundary consists of finitely many closed, nonintersecting Jordan curves. Suppose $A \subset A(G)$ is a $HGQ$-algebra, in which the weak Nullstellensatz holds. Then its stable rank is one.

**Proof.** We have to show that every unimodular vector $(f, g) \in A^2$ is reducible. As before, we may assume that $g \neq 0$.

**Step 1.** $f$ is holomorphic in $\overline{G}$. We argue as in the proof of Theorem 2.2. We have to replace the use of Theorem 2.1 by the use of Theorem 3.1.

**Step 2.** $f \in A$. Since the vector $(f, g)$ is unimodular, there exist $\alpha, \beta \in A$ such that

$$\alpha f + \beta g = 1.$$  

Because $A$ is a $HGQ$-algebra, we can approximate the function $\alpha$ by functions which are holomorphic in $\overline{G}$. Especially $A$ is a $Q$-algebra, so there exists a neighborhood $U$ of 1 such that every member of $U$ is invertible in $A$. Now multiplication by $f$ is a continuous operation on $A^2$, so there exist functions $\tilde{\alpha} \in A$ holomorphic in $\overline{G}$ such that $\tilde{\alpha} f + \beta g =: u$ is invertible in $A$. This identity implies that the vector $(\tilde{\alpha}, g)$ is unimodular, hence reducible by the first step. So we have $h \in A$, $v \in A^x$ such that

$$\tilde{\alpha} + ug = v.$$  

Now we insert this in the equation $\tilde{\alpha} f + \beta g = u$ and are done. \(\Box\)

We will give two examples.

$$\Lambda_{\alpha}(\overline{D}) := \{f \in A(\overline{D}) : f \text{ satisfies a Hölder–Lipschitz condition on } \overline{D} \text{ of order } \alpha\}$$
and
\[ A_0^\infty(\overline{D}) := \{ f \in A^\infty(\overline{D}) : f'(0) = 0 \} . \]
The assumptions for Theorem 3.2 are easily verified, so the stable rank of both algebras is one.

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**REFERENCES**


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