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STABLE RANK OF SUBALGEBRAS OF THE DISC ALGEBRA

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ABSTRACT. We calculate the stable rank of subalgebras (in the algebraic sense) of the disc algebra. We extend the results to subalgebras of $A(\overline{G})$, where G is a bounded domain whose boundary consists of finitely many closed nonintersecting Jordan curves.

INTRODUCTION

The concept of the stable rank of a ring, introduced, by H. Bass [1], has been very useful in treating some problems in algebraic K-theory. In a serial of papers G. Corach and F. D. Suárez calculated the stable rank of Banach algebras of holomorphic functions. Among them are the well-known algebras $A(K)$ of all continuous complex valued functions on a compact set $K \subset \mathbf{C}$ which are analytic in its interior. However, the stable rank of the disc algebra $A(\overline{\mathbf{D}})$ was first calculated by P. Jones, D. Marshall and T. Wolff [8] by an entirely different technique.

In this paper we study the stable rank of subalgebras of the disc algebra $A(\overline{\mathbf{D}})$. Generalizations to subalgebras of $A(\overline{G})$ are given, where G denotes a bounded domain whose boundary consists of finitely many nonintersecting closed Jordan curves. However, the best results are obtained for subalgebras of the disc algebra $A(\overline{\mathbf{D}})$.

Since our algebras need not be normable, our results are generalizations of results of G. Corach and F. D. Suárez, see [3]. Also this paper complements the results in [10].

1. It is well known that the group of units in Banach algebras is open. Unfortunately, this feature is lost in the general case of a topological algebra. This led to the following definitions:

A topological algebra A is called a Q-algebra if the set of units, A^x, is open in A.

In this paper we consider complex, commutative Q-algebras with unit element being denoted by 1.

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Given a Q-algebra A, an element $a \in A^n$ is called *unimodular* if there exists $b \in A^n$ such that

$$\langle b, a \rangle := \sum_{i=1}^{n} b_i a_i = 1 .$$

We denote by $U_n(A)$ the set of unimodular elements of A^n. Finally, $a = (a_1, \dots, a_n) \in U_n(A)$ is called *reducible* if there exists $x_1, \dots, x_{n-1}$ in A such that

$$(a_1 + x_1 a_n, \dots, a_{n-1} + x_{n-1} a_n) \in U_{n-1}(A).$$

The *stable rank* of A, denoted by $\mathrm{sr}(A)$, is the least integer n such that every $a \in U_{n+1}(A)$ is reducible.

From the theory of the stable rank of Q-algebras we mention the following fact; see [9, p. 18, Korollar 1] or [4, Proposition 1]. (The proof given there is also valid in Q-algebras.)

Proposition 1.1. *Suppose that A is a Q-algebra, $\gamma\colon [0,1] \to \mathbf{C}$ is a continuous curve and let $\Gamma\colon [0,1] \to U_2(A)$, $\Gamma(t) := (a - \gamma(t), b)$ such that $(a - \gamma(0), b)$ is reducible. Then $(a - \gamma(1), b)$ is reducible, too.*

2. Let K be a compact set of the plane $\mathbf{C}$ and let $A(K)$ denote the Banach algebra of all complex valued continuous functions which are holomorphic in the interior K^0 of K.

Let A be a subalgebra of $A(K)$ in the algebraic sense. We say that the *Nullstellensatz* holds in A iff the following condition holds for every n: $(f_1, \dots, f_n)$ is unimodular iff the functions $f_1, \dots, f_n$ have no common zero in K. If the last condition is known to hold for $n = 1, 2$, we say that the *weak Nullstellensatz* holds in A. If, finally, this condition is known to be true for $n = 1$ then A is said to be *inversionally closed*. For example, the Nullstellensatz holds in the well-known disc algebra $A(\overline{\mathbf{D}})$.

Suppose now that the weak Nullstellensatz holds in the algebra $A \subset A(K)$ The inversion set $I(f, g)$ of f, $g \in A$ is defined to be the set of all complex numbers a such that the vector $(f - a, g)$ is unimodular, for short,

$$I(f, g) = \{a \in \mathbf{C}\colon (f - a, g) \in U_2(A)\}.$$

Since the weak Nullstellensatz holds in A, we can restate this as follows:

$$\mathbf{C} \setminus I(f, g) = f(Z_g),$$

where Z_g denotes the set of zeros of g in K. This identity shows that every number $\lambda \in \mathbf{C} \setminus f(K)$ is a member of the inversion set of $I(f, g)$, moreover, $(f - \lambda, g)$ is reducible, since the algebra A is inversionally closed.

Remarks. If the weak Nullstellensatz holds in $A \subset A(K)$, then the algebra A, endowed with the topology of uniform convergence, is a Q-algebra.

The author has no example of an algebra A for which the weak Nullstellensatz holds, but the Nullstellensatz does not hold.

In view of Proposition 1.1 the inversion set $I(f, g)$ plays an important role.

In this paragraph, we will consider mainly subalgebras of the algebra $H^{1,1}(\overline{\mathbf{D}})$, where $H^{1,1}(\overline{\mathbf{D}})$ denotes all those functions of the disc algebra $A(\overline{\mathbf{D}})$ whose boundary function is absolutely continuous, i.e.

$$H^{1,1}(\overline{\mathbf{D}}) := \{f \in A(\overline{\mathbf{D}}) \colon f|_{\partial\mathbf{D}} \text{ is absolutely continuous}\}.$$

Further information on $H^{1,1}(\overline{\mathbf{D}})$ is available in [5], Theorem 3.10.

Theorem 2.1. *let A be a subalgebra of $H^{1,1}(\overline{\mathbf{D}})$ in which the weak Nullstellensatz holds, and suppose that (f, g) is unimodular, where g is not the zero function. Then the complement $\mathbf{C} \setminus I(f, g)$ of the inversion set is totally disconnected.*

Proof. Suppose on the contrary that there exists a closed connected subset J of the compact set $\mathbf{C} \setminus I(f, g)$ which contains more than one point. Since the weak Nullstellensatz holds in A, we have

$$\mathbf{C} \setminus I(f, g) = f(Z_g).$$

Moreover, since J does not have isolated points, for every given point $w \in J$ there exists a sequence $(z_n)_{n\in\mathbf{N}}$ such that

$$\lim_{n\to\infty} f(z_n) = w, \qquad g(z_n) = 0 \qquad (n \in \mathbf{N}).$$

The identity theorem for analytic functions implies that every accumulation point of $(z_n)_{n\in\mathbf{N}}$ has unit modulus. So there exists a compact set $E \subset \partial\mathbf{D}$ such that

$$(1) \qquad J = f(E), \qquad g(E) = \{0\}.$$

As a subset of a set of Lebesgue measure zero, E also has Lebesgue measure zero.

Let p_x resp. p_y denote the projection on the real resp. imaginary axis.

Since both projections are continuous and J is connected, the sets $p_x(J)$ and $p_y(J)$ are intervals on the real resp. imaginary axis. On the other hand, they are the absolutely continuous image of the set $E \subset \partial D$ which has Lebesgue measure zero. So they also have Lebesgue measure zero and consist therefore of exactly one point. Thus we have the contradiction that J consists of one point.

This theorem is the main ingredient in proving the following result.

Theorem 2.2. *Let A be a subalgebra of $H^{1,1}(\overline{\mathbf{D}})$ in which the weak Nullstellensatz holds. Then its stable rank is one.*

Remarks. It should be noted that no assumptions on topology for the algebra A are made.

To prove Theorem 2.2 we use the fact that $f(Z_g)$ is totally disconnected. This argument has been used before, implicitly or explicitly, see [2, Theorem 1.13] or [4, Corollary 2].

Proof. We have to show that every unimodular vector (f, g) is reducible. Since the weak Nullstellensatz holds in A, this is equivalent to the existence of a function $h \in A$ such that $f + hg$ is zerofree in $\overline{\mathbf{D}}$. Of course, we can assume that g is not the zero function. By Proposition 1.1 and the subsequent remark it is enough to exhibit a path γ such that $(f - \gamma(t), g)$ is unimodular, $\gamma(0) = 0$ and $(f - \gamma(1), g)$ is reducible. By definition $(f - a, g)$ is unimodular iff $a \in I(f, g)$. Theorem 2.1 implies that $\mathbf{C} \setminus I(f, g)$ is totally disconnected and compact, that is, its (covering) dimension is zero ([7], p. 20, Section A). By a result of plane topology, see [7], Theorem IV.4, the open set $I(f, g)$ is connected (and unbounded). Choose any point $b \notin f(\overline{\mathbf{D}})$ and a path $\gamma \subset I(f, g)$ joining 0 and b. Since A is inversionally closed we have $b \in I(f, g)$. An inspection shows that γ has the required properties. □

This theorem implies that the following algebras have stable rank one:

$$A^N(\overline{\mathbf{D}}) := \{f \in A(\overline{\mathbf{D}}) \colon f^{(n)} \in A(\overline{\mathbf{D}}), n = 1, \dots, N\};$$
$$A^\infty(\overline{\mathbf{D}}) := \{f \in A(\overline{\mathbf{D}}) \colon f^{(n)} \in A(\overline{\mathbf{D}}) \text{ for all numbers } n\}.$$

Note that the topological algebra $A^\infty(\overline{\mathbf{D}})$ is not normable, [11].

3. It is natural to ask whether Theorem 2.1 holds in general domains.

Theorem 3.1. *Let G be a bounded domain whose boundary consists of finitely many nonintersecting closed Jordan curves. Assume further that f is holomorphic in $\overline{G}$ and $g \in A(\overline{G})$ is not identical zero. Then the compact set $f(Z_g)$ is totally disconnected.*

Proof. Suppose on the contrary that there exists a continuum J in $f(Z_g)$. Since J cannot contain isolated points, the identity theorem for holomorphic functions implies the existence of a compact set $K \subset \partial G$ such that

$$J = f(K) \text{ and } g(K) = \{0\}.$$

It is well known from topology that there exist continua $J' \subset J$ with diameter less than a given positive number ε. Since f' is also holomorphic in $\overline{G}$, we can therefore assume that no point of J is a critical point of f, that is, we have

$$J = f(K), \qquad g(K) = \{0\} \qquad \text{and } 0 \notin f'(K).$$

Now the function f is locally injective on K, and a standard argument shows the existence of compact sets K_j, $j = 1, \dots, n$ such that

$$K = \bigcup_{j=1}^{n} K_j \qquad \text{and } f|_{K_j} \text{ is injective}.$$

This yields

$$J = f(K) = \bigcup_{j=1}^{n} f(K_j).$$

Now the set $Z_g \cap \partial G$ is totally disconnected. (A continuum would be on exactly one Jordan curve Γ, so it is an arc. The well-known two-constant theorem (see [6], p. 299, Satz 2) implies the contradiction $g = 0$.

Every compact subset K_j of $Z_g \cap \partial G$ is also totally disconnected. Their (covering) dimension is therefore zero ([7], p. 20, Section A). Since the functions $f|_{K_j}$ are homeomorphisms, the compact sets $f(K_j)$ also have dimension zero.

By the sum theorem in dimension theory J has dimension zero, too ([7], Theorem III.2). Because J is compact, this implies that J is totally disconnected. But by assumption J was a continuum, which is a contradiction. Therefore $f(Z_g)$ is totally disconnected. □

We will use this result to calculate the stable rank of some algebras.

Let K be a compact plane set. We say that a subalgebra A of $A(K)$ is a holomorphically generated Q-algebra, for short, HGQ-algebra iff the following two conditions hold:

(i) A is a Q-algebra.
(ii) There exists a set L of functions holomorphic on K such that $1 \in L$, and in each neighborhood of a suitable element of A we find a member of L, that is the closure of L is A.

For example, the disc algebra $A(\overline{\mathbf{D}})$ is a HGQ-algebra.

Theorem 3.2. *Let G be a bounded domain whose boundary consists of finitely many closed, nonintersecting Jordan curves. Suppose $A \subset A(\overline{G})$ is a HGQ-algebra, in which the weak Nullstellensatz holds. Then its stable rank is one.*

Proof. We have to show that every unimodular vector $(f, g) \in A^2$ is reducible. As before, we may assume that $g \neq 0$.

Step 1. f is holomorphic in $\overline{G}$. We argue as in the proof of Theorem 2.2. We have to replace the use of Theorem 2.1 by the use of Theorem 3.1.

Step 2. $f \in A$. Since the vector (f, g) is unimodular, there exist α, $\beta \in A$ such that

$$\alpha f + \beta g = 1 .$$

Because A is a HGQ-algebra, we can approximate the function α by functions which are holomorphic in $\overline{G}$. Especially A is a Q-algebra, so there exists a neighborhood U of 1 such that every member of U is invertible in A. Now multiplication by f is a continuous operation on A^2, so there exist functions $\tilde{\alpha} \in A$ holomorphic in $\overline{G}$ such that $\tilde{\alpha} f + \beta g =: u$ is invertible in A. This identity implies that the vector $(\tilde{\alpha}, g)$ is unimodular, hence reducible by the first step. So we have $h \in A$, $v \in A^x$ such that

$$\tilde{\alpha} + ug = v .$$

Now we insert this in the equation $\tilde{\alpha} f + \beta g = u$ and are done. □

We will give two examples.

$$\Lambda_\alpha(\overline{\mathbf{D}}) := \{f \in A(\overline{\mathbf{D}}) : f \text{ satisfies a Hölder–Lipschitz condition on } \overline{\mathbf{D}} \text{ of order } \alpha\}$$

and

$$A_0^\infty(\overline{\mathbf{D}}) := \{f \in A^\infty(\overline{\mathbf{D}}) : f'(0) = 0\}.$$

The assumptions for Theorem 3.2 are easily verified, so the stable rank of both algebras is one.

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