

## A PROPERTY OF INFINITELY DIFFERENTIABLE FUNCTIONS

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**ABSTRACT.** The existence of  $\lim_{n \rightarrow \infty} \|f^{(n)}\|_p^{1/n}$  for an arbitrary function  $f(x) \in C^\infty(\mathbf{R})$  such that  $f^{(n)}(x) \in L^p(\mathbf{R})$ ,  $n = 0, 1, \dots$  ( $1 \leq p \leq \infty$ ) and the concrete calculation of  $\lim_{n \rightarrow \infty} \|f^{(n)}\|_p^{1/n}$  are shown.

**Theorem 1.** *Let  $1 \leq p \leq \infty$  and  $f(x) \in C^\infty(\mathbf{R})$  such that  $f^{(n)}(x) \in L^p(\mathbf{R})$ ,  $n = 0, 1, \dots$ . Then there always exists the limit*

$$d_f = \lim_{n \rightarrow \infty} \|f^{(n)}\|_p^{1/n},$$

and moreover

$$d_f = \sigma_f = \sup\{|\xi| : \xi \in \text{supp } \tilde{f}(\xi)\},$$

where the last equality is the definition of  $\sigma_f$  and  $\tilde{f}(\xi)$  is the Fourier transform of the function  $f(x)$ . \*

*Proof.* We shall begin by showing that there exists the limit

$$(1) \quad d_f = \lim_{n \rightarrow \infty} \|f^{(n)}\|_p^{1/n}.$$

Without loss of generality we may assume that  $\|f\|_p = 1$ . Then using the Kolmogoroff-Stein theorem [1, 2], we have

$$\|f^{(k)}\|_p^n \leq (\pi/2)^n \|f^{(n)}\|_p^k, \quad 0 < k < n,$$

for any  $n = 2, 3, \dots$ , and hence

$$(2) \quad \|f^{(k)}\|_p^{1/k} \leq (\pi/2)^{1/k} \|f^{(n)}\|_p^{1/n}, \quad 0 < k < n.$$

By (2) it follows that

$$\|f^{(k)}\|_p^{1/k} \leq (\pi/2)^{1/k} \lim_{n \rightarrow \infty} \|f^{(n)}\|_p^{1/n}$$

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\* The Fourier transform is in the sense of [3].

for any  $k = 1, 2, \dots$ ; therefore

$$(3) \quad \overline{\lim}_{k \rightarrow \infty} \|f^{(k)}\|_p^{1/k} \leq \underline{\lim}_{n \rightarrow \infty} \|f^{(n)}\|_p^{1/n}.$$

Equation (1) is immediate from (3).

Further, we shall prove that  $d_f = \sigma_f$ . We first observe that

$$(4) \quad d_f \leq \sigma_f.$$

It is enough to show (4) for  $\sigma_f < \infty$ . Then using  $f \in \mathcal{S}'$  (this follows from  $f \in L^p(\mathbf{R})$ ) and the well-known Paley-Wiener-Schwartz theorem, we obtain that  $f$  is an analytic function of exponential type  $\leq \sigma_f$ . Hence by the Bernstein-Nikolsky inequality [3, p. 115] it follows that

$$\|f^{(n)}\|_p \leq \sigma_f^n \|f\|_p, \quad n = 0, 1, \dots,$$

and (4) is an immediate consequence of the last inequalities.

Finally, we claim that  $d_f \geq \sigma_f$ . We divide the proof into two cases.

Case 1.  $p = \infty$ . Assume the contrary, that  $d_f < \sigma_f$ . Then there exist numbers  $M < \infty, \sigma < \sigma_f$  such that

$$\|f^{(n)}\|_\infty \leq M\sigma^n, \quad n = 0, 1, \dots$$

Therefore, using the inverse theorem of Bernstein we have that  $f$  is an analytic function of exponential type  $\leq \sigma < \infty$ . Consequently, it follows from Schwartz's theorem [3, p. 110] that  $\text{supp } \tilde{f}(\xi) \subset \{\xi: |\xi| \leq \sigma\}$ . This contradicts the assumption that  $\sigma < \sigma_f$ .

Case 2.  $1 \leq p < \infty$ . Let

$$(5) \quad f_k(x) = k \int_0^{1/k} f(x+t) dt, \quad k = 1, 2, \dots$$

Then by Jensen's inequality we obtain

$$|f_k^{(n)}(x)|^p \leq k \int_0^{1/k} |f^{(n)}(x+t)|^p dt, \quad k = 1, 2, \dots,$$

for any  $n = 0, 1, \dots$ ; therefore,

$$(6) \quad \|f_k^{(n)}\|_\infty \leq k^{1/p} \|f^{(n)}\|_p, \quad n = 0, 1, \dots, k = 1, 2, \dots$$

On the other hand, Case 1 shows that

$$(7) \quad \sigma_{f_k} = \lim_{n \rightarrow \infty} \|f_k^{(n)}\|_\infty^{1/n}, \quad k = 1, 2, \dots$$

Combining (6) and (7) yields

$$\sigma_{f_k} \leq \lim_{n \rightarrow \infty} \|f^{(n)}\|_p^{1/n} = d_f, \quad k = 1, 2, \dots$$

Consequently, to complete the proof it remains to show that

$$\sigma_f \leq \underline{\lim}_{k \rightarrow \infty} \sigma_{f_k}$$

and therefore the problem is now reduced to proving that

$$(8) \quad |\xi| \leq \liminf_{k \rightarrow \infty} \sigma_{f_k}$$

for any point  $\xi \in \text{supp } \tilde{f}(\xi)$ .

Assume the contrary, that (8) is not satisfied. Then there exist a point  $\xi_0 \in \text{supp } \tilde{f}(\xi)$ , a number  $\varepsilon_0 > 0$ , and a subsequence  $\{k_m\}$  (for simplicity of notation we assume that  $\xi_0 > 0$ ,  $k_m = m$ ,  $m = 1, 2, \dots$ ) such that

$$(9) \quad \sigma_{f_m} \leq \xi_0 - 2\varepsilon_0, \quad m = 1, 2, \dots$$

On the other hand, it is well known that

$$(10) \quad \|f(x+y) - f(x)\|_p \rightarrow 0, \quad |y| \rightarrow 0.$$

It obviously follows from (5) and (10) that

$$\|f_k - f\|_p \rightarrow 0, \quad k \rightarrow \infty;$$

therefore,  $f_k$  converges weakly to  $f$  in  $\mathcal{S}'$ , and therefore  $\tilde{f}_k$  also converges weakly to  $\tilde{f}$  in  $\mathcal{S}'$ .

Now we choose a function  $\varphi(x) \in C_0^\infty(\mathbf{R})$  such that  $\langle \tilde{f}, \varphi \rangle \neq 0$ ,  $\text{supp } \varphi(x) \subset [\xi_0 - \varepsilon_0, \xi_0 + \varepsilon_0]$ . Then it follows readily from  $\tilde{f}_m \rightarrow \tilde{f}$  weakly in  $\mathcal{S}'$  and (9) that

$$0 = \langle \tilde{f}_m, \varphi \rangle \rightarrow \langle \tilde{f}, \varphi \rangle \neq 0, \quad m \rightarrow \infty.$$

We thus arrive at a contradiction. The proof is complete.

We close this paper with the following

**Theorem 2.** *Suppose that  $f(x) \in C^\infty(\mathbf{R})$  is an arbitrary  $2\pi$ -periodic function and  $1 \leq p \leq \infty$ . Then there exists the limit*

$$d_f = \lim_{n \rightarrow \infty} \| \|f^{(n)}\| \|_p^{1/n},$$

and moreover

$$d_f = \sigma_f = \sup\{|k| : k \in \text{supp } \tilde{f}(\xi)\},$$

where  $\| \cdot \|_p$  is the  $L^p(0, 2\pi)$ -norm.

*Proof.* Representing the function  $f(x)$  by its Fourier series, we have

$$f(x) = \sum_{k=-\infty}^{\infty} f_k \exp(ikx),$$

where

$$f_k = (2\pi)^{-1} \langle f, \exp(-ikx) \rangle, \quad k = 0, \pm 1, \dots$$

Therefore,

$$f^{(n)}(x) = \sum_{k=-\infty}^{\infty} f_k (ik)^n \exp(ikx), \quad n = 0, 1, \dots$$

Hence, in view of the Hölder inequality,

$$\begin{aligned} |f_k k^n| &= (2\pi)^{-1} |(f^{(n)}, \exp(-ikx))| \\ &\leq (2\pi)^{-1/p} |||f^{(n)}|||_p, \end{aligned}$$

where  $n = 0, 1, \dots$ ;  $k = 0, \pm 1, \dots$ .

Consequently,

$$(11) \quad \lim_{n \rightarrow \infty} |f_k k^n|^{1/n} = |k| \leq \underline{\lim}_{n \rightarrow \infty} |||f^{(n)}|||_p^{1/n}$$

for any index  $k$  such that  $f_k \neq 0$ .

Using

$$\tilde{f}(\xi) = \sum_{k=-\infty}^{\infty} f_k \delta(\xi + k)$$

and (11), we have

$$(12) \quad \sigma_f \leq \underline{\lim}_{n \rightarrow \infty} |||f^{(n)}|||_p^{1/n}.$$

Further, we show that

$$(13) \quad \overline{\lim}_{n \rightarrow \infty} |||f^{(n)}|||_p^{1/n} \leq \sigma_f.$$

It is enough to prove (13) for  $\sigma_f < \infty$ . Then by the Paley-Wiener-Schwartz theorem it follows that  $f$  is an analytic function of exponential type  $\leq \sigma_f$ . Hence, it follows from the inequality of Bernstein and Nikolsky that

$$|||f^{(n)}|||_p \leq \sigma_f^n |||f|||_p, \quad n = 0, 1, \dots,$$

and (13) is an immediate consequence of the last inequalities.

Combining (12) and (13) yields

$$\underline{\lim}_{n \rightarrow \infty} |||f^{(n)}|||_p^{1/n} = \overline{\lim}_{n \rightarrow \infty} |||f^{(n)}|||_p^{1/n} = \sigma_f.$$

The theorem is proved.

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