**A PROPERTY OF INFINITELY DIFFERENTIABLE FUNCTIONS**

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**Abstract.** The existence of \( \lim_{n \to \infty} \| f^{(n)} \|_p^{1/n} \) for an arbitrary function \( f(x) \in C^\infty(\mathbb{R}) \) such that \( f^{(n)}(x) \in L^p(\mathbb{R}), \ n = 0, 1, \ldots \) \( (1 \leq p \leq \infty) \) and the concrete calculation of \( \lim_{n \to \infty} \| f^{(n)} \|_p^{1/n} \) are shown.

**Theorem 1.** Let \( 1 \leq p \leq \infty \) and \( f(x) \in C^\infty(\mathbb{R}) \) such that \( f^{(n)}(x) \in L^p(\mathbb{R}), \ n = 0, 1, \ldots \). Then there always exists the limit

\[
\frac{df}{dx} = \lim_{n \to \infty} \| f^{(n)} \|_p^{1/n},
\]

and moreover

\[
d_f = \sigma_f = \sup \{|\xi| : \xi \in \text{supp} \tilde{f}(\xi)\},
\]

where the last equality is the definition of \( \sigma_f \) and \( \tilde{f}(\xi) \) is the Fourier transform of the function \( f(x) \).

**Proof.** We shall begin by showing that there exists the limit

\[
(1) \quad \frac{df}{dx} = \lim_{n \to \infty} \| f^{(n)} \|_p^{1/n}.
\]

Without loss of generality we may assume that \( \| f \|_p = 1 \). Then using the Kolmogoroff-Stein theorem [1, 2], we have

\[
\| f^{(k)} \|_p \leq (\pi/2)^n \| f^{(n)} \|_p, \quad 0 < k < n,
\]

for any \( n = 2, 3, \ldots, \) and hence

\[
(2) \quad \| f^{(k)} \|_p^{1/k} \leq (\pi/2)^{1/k} \| f^{(n)} \|_p^{1/n}, \quad 0 < k < n.
\]

By (2) it follows that

\[
\| f^{(k)} \|_p^{1/k} \leq (\pi/2)^{1/k} \lim_{n \to \infty} \| f^{(n)} \|_p^{1/n}
\]

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* The Fourier transform is in the sense of [3].
for any \( k = 1, 2, \ldots \); therefore

\[
\lim_{k \to \infty} \|f^{(k)}\|_p^{1/k} \leq \lim_{n \to \infty} \|f^{(n)}\|_p^{1/n}.
\]

Equation (1) is immediate from (3).

Further, we shall prove that \( d_f = \sigma_f \). We first observe that

\[
d_f \leq \sigma_f.
\]

It is enough to show (4) for \( \sigma_f < \infty \). Then using \( f \in \mathcal{S}' \) (this follows from \( f \in L^p(\mathbb{R}) \)) and the well-known Paley-Wiener-Schwartz theorem, we obtain that \( f \) is an analytic function of exponential type \( \leq \sigma_f \). Hence by the Bernstein-Nikolsky inequality [3, p. 115] it follows that

\[
\|f^{(n)}\|_p \leq \sigma_f^n \|f\|_p, \quad n = 0, 1, \ldots,
\]

and (4) is an immediate consequence of the last inequalities.

Finally, we claim that \( d_f \geq \sigma_f \). We divide the proof into two cases.

Case 1. \( p = \infty \). Assume the contrary, that \( d_f < \sigma_f \). Then there exist numbers \( M < \infty, \sigma < \sigma_f \) such that

\[
\|f^{(n)}\|_\infty \leq M\sigma^n, \quad n = 0, 1, \ldots.
\]

Therefore, using the inverse theorem of Bernstein we have that \( f \) is an analytic function of exponential type \( \leq \sigma < \infty \). Consequently, it follows from Schwartz’s theorem [3, p. 110] that \( \text{supp} \tilde{f}(\xi) \subset \{\xi: |\xi| \leq \sigma\} \). This contradicts the assumption that \( \sigma < \sigma_f \).

Case 2. \( 1 < p < \infty \). Let

\[
f_k(x) = k \int_0^{1/k} f(x + t) \, dt, \quad k = 1, 2, \ldots.
\]

Then by Jensen’s inequality we obtain

\[
|f_k^{(n)}(x)|^p \leq k \int_0^{1/k} |f^{(n)}(x + t)|^p \, dt, \quad k = 1, 2, \ldots,
\]

for any \( n = 0, 1, \ldots \); therefore,

\[
\|f_k^{(n)}\|_\infty \leq k^{1/p} \|f^{(n)}\|_p, \quad n = 0, 1, \ldots k = 1, 2, \ldots.
\]

On the other hand, Case 1 shows that

\[
\sigma_{f_k} = \lim_{n \to \infty} \|f_k^{(n)}\|_\infty^{1/n}, \quad k = 1, 2, \ldots.
\]

Combining (6) and (7) yields

\[
\sigma_{f_k} \leq \lim_{n \to \infty} \|f^{(n)}\|_p^{1/n} = d_f, \quad k = 1, 2, \ldots.
\]

Consequently, to complete the proof it remains to show that

\[
\sigma_f \leq \lim_{k \to \infty} \sigma_{f_k}
\]
and therefore the problem is now reduced to proving that

\[ |\xi| \leq \lim_{k \to \infty} \sigma_{f_k} \]

for any point \( \xi \in \text{supp} \hat{f}(\xi) \).

Assume the contrary, that (8) is not satisfied. Then there exist a point \( \xi_0 \in \text{supp} \hat{f}(\xi) \), a number \( \varepsilon_0 > 0 \), and a subsequence \( \{k_m\} \) (for simplicity of notation we assume that \( \xi_0 > 0 \), \( k_m = m \), \( m = 1, 2, \ldots \)) such that

\[ \sigma_{f_m} \leq \xi_0 - 2\varepsilon_0, \quad m = 1, 2, \ldots. \]

On the other hand, it is well known that

\[ \|f(x + y) - f(x)\|_p \to 0, \quad |y| \to 0. \]

It obviously follows from (5) and (10) that

\[ \|f_k - f\|_p \to 0, \quad k \to \infty; \]

therefore, \( f_k \) converges weakly to \( f \) in \( S' \), and therefore \( \hat{f}_k \) also converges weakly to \( \hat{f} \) in \( S' \).

Now we choose a function \( \varphi(x) \in C^\infty_0(\mathbb{R}) \) such that \( \langle \hat{f}, \varphi \rangle \neq 0 \), \( \text{supp} \varphi(x) \subseteq [\xi_0 - \varepsilon_0, \xi_0 + \varepsilon_0] \). Then it follows readily from \( \hat{f}_m \to \hat{f} \) weakly in \( S' \) and (9) that

\[ 0 = \langle \hat{f}_m, \varphi \rangle \to \langle \hat{f}, \varphi \rangle \neq 0, \quad m \to \infty. \]

We thus arrive at a contradiction. The proof is complete.

We close this paper with the following

**Theorem 2.** Suppose that \( f(x) \in C^\infty(\mathbb{R}) \) is an arbitrary \( 2\pi \)-periodic function and \( 1 \leq p \leq \infty \). Then there exists the limit

\[ d_f = \lim_{n \to \infty} \|f^{(n)}\|_p^{1/n}, \]

and moreover

\[ d_f = \sigma_f = \sup\{|k|: k \in \text{supp} \hat{f}(\xi)\}, \]

where \( \|\cdot\|_p \) is the \( L^p(0, 2\pi) \)-norm.

**Proof.** Representing the function \( f(x) \) by its Fourier series, we have

\[ f(x) = \sum_{k = -\infty}^{\infty} f_k \exp(ikx), \]

where

\[ f_k = (2\pi)^{-1} (f, \exp(-ikx)), \quad k = 0, \pm 1, \ldots. \]

Therefore,

\[ f^{(n)}(x) = \sum_{k = -\infty}^{\infty} f_k(ik)^n \exp(ikx), \quad n = 0, 1, \ldots. \]
Hence, in view of the H"older inequality,
\[ |f_k n| = (2\pi)^{-1} |(f^{(n)} \exp(-ikx))| \leq (2\pi)^{1/p} \|f^{(n)}\|_p, \]
where $n = 0, 1, \ldots$; $k = 0, \pm 1, \ldots$.

Consequently,
\[ \lim_{n\to\infty} |f_k n|^{1/n} = |k| \leq \lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} \]
for any index $k$ such that $f_k \neq 0$.

Using
\[ \hat{f}(\xi) = \sum_{k=-\infty}^{\infty} f_k \delta(\xi + k) \]
and (11), we have
\[ \sigma_f \leq \lim_{n\to\infty} \|f^{(n)}\|_p^{1/n}. \]

Further, we show that
\[ \lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} \leq \sigma_f. \]
It is enough to prove (13) for $\sigma_f < \infty$. Then by the Paley-Wiener-Schwartz theorem it follows that $f$ is an analytic function of exponential type $\leq \sigma_f$. Hence, it follows from the inequality of Bernstein and Nikolsky that
\[ \|f^{(n)}\|_p \leq \sigma_f^n \|f\|_p, \quad n = 0, 1, \ldots, \]
and (13) is an immediate consequence of the last inequalities.

Combining (12) and (13) yields
\[ \lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = \lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = \sigma_f. \]

The theorem is proved.

**References**


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