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A PROPERTY OF INFINITELY DIFFERENTIABLE FUNCTIONS

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ABSTRACT. The existence of $\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n}$ for an arbitrary function $f(x) \in C^\infty(\mathbf{R})$ such that $f^{(n)}(x) \in L^p(\mathbf{R})$, $n = 0, 1, \dots$ $(1 \le p \le \infty)$ and the concrete calculation of $\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n}$ are shown.

Theorem 1. *Let* $1 \le p \le \infty$ *and* $f(x) \in C^\infty(\mathbf{R})$ *such that* $f^{(n)}(x) \in L^p(\mathbf{R})$, $n = 0, 1, \dots$. *Then there always exists the limit*

$$d_f = \lim_{n\to\infty} \|f^{(n)}\|_p^{1/n},$$

and moreover

$$d_f = \sigma_f = \sup\{|\xi| : \xi \in \operatorname{supp} \tilde{f}(\xi)\},$$

where the last equality is the definition of σ_f *and* $\tilde{f}(\xi)$ *is the Fourier transform of the function* $f(x)$.*

Proof. We shall begin by showing that there exists the limit

$$d_f = \lim_{n\to\infty} \|f^{(n)}\|_p^{1/n}. \tag{1}$$

Without loss of generality we may assume that $\|f\|_p = 1$. Then using the Kolmogoroff-Stein theorem [1, 2], we have

$$\|f^{(k)}\|_p^n \le (\pi/2)^n \|f^{(n)}\|_p^k, \qquad 0 < k < n,$$

for any $n = 2, 3, \dots$, and hence

$$\|f^{(k)}\|_p^{1/k} \le (\pi/2)^{1/k} \|f^{(n)}\|_p^{1/n}, \qquad 0 < k < n. \tag{2}$$

By (2) it follows that

$$\|f^{(k)}\|_p^{1/k} \le (\pi/2)^{1/k} \varliminf_{n\to\infty} \|f^{(n)}\|_p^{1/n}$$

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* The Fourier transform is in the sense of [3].

for any $k = 1, 2, \dots$; therefore

(3) $$\overline{\lim_{k\to\infty}} \|f^{(k)}\|_p^{1/k} \le \underline{\lim}_{n\to\infty} \|f^{(n)}\|_p^{1/n}.$$

Equation (1) is immediate from (3).

Further, we shall prove that $d_f = \sigma_f$. We first observe that

(4) $$d_f \le \sigma_f.$$

It is enough to show (4) for $\sigma_f < \infty$. Then using $f \in \mathscr{S}'$ (this follows from $f \in L^p(\mathbf{R})$) and the well-known Paley-Wiener-Schwartz theorem, we obtain that f is an analytic function of exponential type $\le \sigma_f$. Hence by the Bernstein–Nikolsky inequality [3, p. 115] it follows that

$$\|f^{(n)}\|_p \le \sigma_f^n \|f\|_p, \qquad n = 0, 1, \dots,$$

and (4) is an immediate consequence of the last inequalities.

Finally, we claim that $d_f \ge \sigma_f$. We divide the proof into two cases.

Case 1. $p = \infty$. Assume the contrary, that $d_f < \sigma_f$. Then there exist numbers $M < \infty$, $\sigma < \sigma_f$ such that

$$\|f^{(n)}\|_\infty \le M\sigma^n, \qquad n = 0, 1, \dots.$$

Therefore, using the inverse theorem of Bernstein we have that f is an analytic function of exponential type $\le \sigma < \infty$. Consequently, it follows from Schwartz's theorem [3, p. 110] that $\operatorname{supp} \tilde{f}(\xi) \subset \{\xi : |\xi| \le \sigma\}$. This contradicts the assumption that $\sigma < \sigma_f$.

Case 2. $1 \le p < \infty$. Let

(5) $$f_k(x) = k \int_0^{1/k} f(x+t)\,dt, \qquad k = 1, 2, \dots.$$

Then by Jensen's inequality we obtain

$$|f_k^{(n)}(x)|^p \le k \int_0^{1/k} |f^{(n)}(x+t)|^p\,dt, \qquad k = 1, 2, \dots,$$

for any $n = 0, 1, \dots$; therefore,

(6) $$\|f_k^{(n)}\|_\infty \le k^{1/p} \|f^{(n)}\|_p, \qquad n = 0, 1, \dots\ k = 1, 2, \dots.$$

On the other hand, Case 1 shows that

(7) $$\sigma_{f_k} = \lim_{n\to\infty} \|f_k^{(n)}\|_\infty^{1/n}, \qquad k = 1, 2, \dots.$$

Combining (6) and (7) yields

$$\sigma_{f_k} \le \lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = d_f, \qquad k = 1, 2, \dots.$$

Consequently, to complete the proof it remains to show that

$$\sigma_f \le \underline{\lim}_{k\to\infty} \sigma_{f_k}$$

and therefore the problem is now reduced to proving that

$$(8) \qquad |\xi| \leq \varliminf_{k\to\infty} \sigma_{f_k}$$

for any point $\xi \in \operatorname{supp} \tilde{f}(\xi)$.

Assume the contrary, that (8) is not satisfied. Then there exist a point $\xi_0 \in \operatorname{supp} \tilde{f}(\xi)$, a number $\varepsilon_0 > 0$, and a subsequence $\{k_m\}$ (for simplicity of notation we assume that $\xi_0 > 0$, $k_m = m$, $m = 1, 2, \dots$) such that

$$(9) \qquad \sigma_{f_m} \leq \xi_0 - 2\varepsilon_0, \qquad m = 1, 2, \dots.$$

On the other hand, it is well known that

$$(10) \qquad \|f(x+y) - f(x)\|_p \to 0, \qquad |y| \to 0.$$

It obviously follows from (5) and (10) that

$$\|f_k - f\|_p \to 0, \qquad k \to \infty;$$

therefore, f_k converges weakly to f in $\mathscr{S}'$, and therefore $\tilde{f}_k$ also converges weakly to $\tilde{f}$ in $\mathscr{S}'$.

Now we choose a function $\varphi(x) \in C_0^\infty(\mathbf{R})$ such that $\langle \tilde{f}, \varphi \rangle \neq 0$, $\operatorname{supp} \varphi(x) \subset [\xi_0 - \varepsilon_0, \xi_0 + \varepsilon_0]$. Then it follows readily from $\tilde{f}_m \to \tilde{f}$ weakly in $\mathscr{S}'$ and (9) that

$$0 = \langle \tilde{f}_m, \varphi \rangle \to \langle \tilde{f}, \varphi \rangle \neq 0, \qquad m \to \infty.$$

We thus arrive at a contradiction. The proof is complete.

We close this paper with the following

Theorem 2. *Suppose that $f(x) \in C^\infty(\mathbf{R})$ is an arbitrary 2π-periodic function and $1 \leq p \leq \infty$. Then there exists the limit*

$$d_f = \lim_{n\to\infty} |||f^{(n)}|||_p^{1/n},$$

and moreover

$$d_f = \sigma_f = \sup\{|k| : k \in \operatorname{supp} \tilde{f}(\xi)\},$$

where $|||\cdot|||_p$ is the $L^p(0, 2\pi)$-norm.

Proof. Representing the function $f(x)$ by its Fourier series, we have

$$f(x) = \sum_{k=-\infty}^{\infty} f_k \exp(ikx),$$

where

$$f_k = (2\pi)^{-1}(f, \exp(-ikx)), \qquad k = 0, \pm 1, \dots.$$

Therefore,

$$f^{(n)}(x) = \sum_{k=-\infty}^{\infty} f_k (ik)^n \exp(ikx), \qquad n = 0, 1, \dots.$$

Hence, in view of the Hölder inequality,

$$\begin{aligned}|f_k k^n| &= (2\pi)^{-1}|(f^{(n)}, \exp(-ikx))| \\ &\le (2\pi)^{-1/p}|||f^{(n)}|||_p,\end{aligned}$$

where $n = 0, 1, \dots ;\ k = 0, \pm 1, \dots$.

Consequently,

$$\lim_{n\to\infty} |f_k k^n|^{1/n} = |k| \le \underline{\lim}_{n\to\infty} |||f^{(n)}|||_p^{1/n} \tag{11}$$

for any index k such that $f_k \neq 0$.

Using

$$\tilde{f}(\xi) = \sum_{k=-\infty}^{\infty} f_k \delta(\xi + k)$$

and (11), we have

$$\sigma_f \le \underline{\lim}_{n\to\infty} |||f^{(n)}|||_p^{1/n}. \tag{12}$$

Further, we show that

$$\overline{\lim}_{n\to\infty} |||f^{(n)}|||_p^{1/n} \le \sigma_f. \tag{13}$$

It is enough to prove (13) for $\sigma_f < \infty$. Then by the Paley-Wiener-Schwartz theorem it follows that f is an analytic function of exponential type $\le \sigma_f$. Hence, it follows from the inequality of Bernstein and Nikolsky that

$$|||f^{(n)}|||_p \le \sigma_f^n |||f|||_p, \qquad n = 0, 1, \dots,$$

and (13) is an immediate consequence of the last inequalities.

Combining (12) and (13) yields

$$\underline{\lim}_{n\to\infty} |||f^{(n)}|||_p^{1/n} = \overline{\lim}_{n\to\infty} |||f^{(n)}|||_p^{1/n} = \sigma_f.$$

The theorem is proved.

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