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POSITIVE SOLUTIONS OF DIFFERENCE EQUATIONS

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ABSTRACT. Consider the difference equation

$$(-1)^{m+1}\Delta^m A_n + \sum_{k=0}^{\infty} p_k A_{n-l_k} = 0, \tag{E}$$

where m is a positive integer, $(p_k)_{k\geq 0}$ is a sequence of positive real numbers and $(l_k)_{k\geq 0}$ is a sequence of integers with $0 \leq l_0 < l_1 < l_2 < \cdots$. The characteristic equation of (E) is

$$-(1-\lambda)^m + \sum_{k=0}^{\infty} p_k \lambda^{-l_k} = 0. \tag{$*$}$$

We prove the following theorem.

Theorem. (i) *For m even, (E) has a positive solution $(A_n)_{n\in Z}$ with $\limsup_{n\to\infty} A_n < \infty$ if and only if $(*)$ has a root in $(0,1)$.*

(ii) *For m odd, (E) has a positive solution $(A_n)_{n\in Z}$ if and only if $(*)$ has a root in $(0,1)$.*

1. INTRODUCTION

Recently, there has been a lot of activity concerning the oscillatory behavior of the solutions of difference equations. See, for example, [1,2,3 and 4] and the references cited therein. Our aim in this paper is to obtain necessary and sufficient conditions for the existence of positive solutions of certain difference equations.

Let $Z = \{\ldots, -1, 0, 1, \ldots\}$. The forward difference operator Δ is defined as usual, i.e.

$$\Delta S_n = S_{n+1} - S_n, \qquad n \in Z$$

for any sequence $(S_n)_{n\in Z}$ of real numbers. Moreover, if $(A_n)_{n\in Z}$ is a sequence, we define

$$\Delta^0 A_n = A_n, \qquad \text{and } \Delta^i A_n = \Delta(\Delta^{i-1} A_n) \qquad (i = 1, 2, \ldots)$$

for every $n \in Z$.

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where m *is a positive integer,* $(p_k)_{k\geq 0}$ *is a sequence of positive real numbers and* $(l_k)_{k\geq 0}$ *is a sequence of integers with* $0 \leq l_0 < l_1 < l_2 < \cdots$.

By a *solution* of (E) we mean a sequence $(A_n)_{n\in Z}$ which satisfies (E) for all $n \in Z$. A solution $(A_n)_{n\in Z}$ of (E) is called *positive* if $A_n > 0$ for every $n \in Z$. Moreover, a positive solution $(A_n)_{n\in Z}$ of (E) is said to be *bounded at* ∞ if $\limsup_{n\to\infty} A_n < \infty$.

The characteristic equation of (E) is

$$-(1-\lambda)^m + \sum_{k=0}^{\infty} p_\kappa \lambda^{-l_k} = 0. \tag{$*$}$$

In this paper we prove the following result.

Theorem. (i) *For* m *even,* (E) *has a positive solution which is bounded at* ∞ *if and only if* $(*)$ *has a root in* $(0, 1)$.

(ii) *For* m *odd,* (E) *has a positive solution if and only if* $(*)$ *has a root in* $(0, 1)$.

To prove our theorem we need two lemmas. These lemmas are established in §2. The proof of the theorem will be given in §3.

It is easy to verify that

$$\sup_{\lambda\in(0,1)} [(1-\lambda)^m \lambda^l] = \frac{m^m l^l}{(m+l)^{m+l}} \qquad \text{for } l \in \{0, 1, \dots\}.$$

(Here, we use the convention that $0^0 = 1$.) Hence, for every $\lambda \in (0, 1)$

$$\begin{aligned} -(1-\lambda)^m + \sum_{k=0}^{\infty} p_k \lambda^{-l_k} &= (1-\lambda)^m \left[-1 + \sum_{k=0}^{\infty} p_k \frac{1}{(1-\lambda)^m \lambda^{l_k}}\right] \\ &\geq (1-\lambda)^m \left[-1 + \sum_{k=0}^{\infty} p_k \frac{(m+l_k)^{m+l_k}}{m^m l_k^{l_k}}\right] \end{aligned}$$

and so the assumption

$$\sum_{k=0}^{\infty} p_k \frac{(m+l_k)^{m+l_k}}{m^m l_k^{l_k}} > 1 \tag{C}$$

implies that $(*)$ has no roots in $(0, 1)$. Therefore, our theorem leads to the following corollary.

Corollary. *Suppose that* (C) *holds. Then*:

(i) *For* m *even, there is no positive solution of* (E) *which is bounded at* ∞.

(ii) *For* m *odd, there is no positive solution of* (E).

The oscillatory behavior of solutions of the difference equation

$$(-1)^{m+1}\Delta^m A_n + \sum_{k=0}^{N} p_k A_{n-l_k} = 0,$$

where N is a non-negative integer, p_k $(k = 0, 1, \dots, N)$ are positive constants and l_k $(k = 0, 1, \dots, N)$ are integers with $0 \le l_0 < l_1 < \cdots < l_N$, can be studied by a detailed analysis of the representation of the solutions in terms of the roots of the characteristic equation. The above equation is the discrete version of the delay differential equation

$$(-1)^{m+1} x^{(m)}(t) + \sum_{k=0}^{N} p_k x(t - \tau_k) = 0,$$

where $N \ge 0$ is an integer, the coefficients $p_0, p_1, \dots, p_N$ are positive numbers and the delays are constants such that $0 \le \tau_0 < \tau_1 < \cdots < \tau_N$. The oscillation of solutions of this differential equation is treated in [5]. However, this is the first paper dealing with difference equations of the form (E). For such difference equations no representation of solutions in terms of the roots of $(*)$ is known.

2. Lemmas

The following two lemmas will be useful in §3.

Lemma 1. *Let $(A_n)_{n\in Z}$ be a positive solution of (E) which is bounded at ∞. Then*

$$(-1)^j \Delta^j A_n > 0 \qquad \textit{for all } n \in Z \quad (j = 0, 1, \dots, m-1, m).$$

Proof. From (E) we obtain for $n \in Z$

$$(-1)^m \Delta^m A_n = \sum_{k=0}^{\infty} p_k A_{n-l_k}$$

and consequently

(1) $$(-1)^m \Delta^m A_n > 0 \qquad \text{for all } n \in Z.$$

For $m = 1$ the proof is complete. So, we assume that $m > 1$. We now claim that

(2) $$(-1)^{m-1} \Delta^{m-1} A_n > 0 \qquad \text{for every } n \in Z.$$

Otherwise, there exists an integer n_1 with

$$(-1)^{m-1} (\Delta^{m-1} A_n)_{n=n_1} \le 0.$$

From (1) it follows that the sequence $((-1)^{m-1}\Delta^{m-1}A_n)_{n\in Z}$ is strictly decreasing. Hence, if we choose an integer $n_2 > n_1$, then we derive for $n \ge n_2$

$$(-1)^{m-1}\Delta^{m-1}A_n \le (-1)^{m-1}(\Delta^{m-1}A_n)_{n=n_2} < (-1)^{m-1}(\Delta^{m-1}A_n)_{n=n_1} \le 0.$$

Therefore

$$(-1)^{m-1}\Delta^{m-1}A_n \le -\gamma \qquad \text{for every } n \ge n_2,$$

where $\gamma = -(-1)^{m-1}(\Delta^{m-1}A_n)_{n=n_2} > 0$. So, we obtain for $n > n_2$

$$\begin{aligned}
(-1)^{m-1}\Delta^{m-2}A_n &- (-1)^{m-1}(\Delta^{m-2}A_n)_{n=n_2} \\
&= [(-1)^{m-1}\Delta^{m-2}A_n - (-1)^{m-1}\Delta^{m-2}A_{n-1}] \\
&\quad + [(-1)^{m-1}\Delta^{m-2}A_{n-1} - (-1)^{m-1}\Delta^{m-2}A_{n-2}] \\
&\quad + \\
&\quad \vdots \\
&\quad + [(-1)^{m-1}(\Delta^{m-2}A_n)_{n=n_2+1} - (-1)^{m-1}(\Delta^{m-2}A_n)_{n=n_2}] \\
&= (-1)^{m-1}\Delta^{m-1}A_{n-1} + (-1)^{m-1}\Delta^{m-1}A_{n-2} + \cdots \\
&\qquad\qquad \cdots + (-1)^{m-1}(\Delta^{m-1}A_n)_{n=n_2} \\
&\leq -\gamma(n - n_2),
\end{aligned}$$

which gives

$$\lim_{n\to\infty}(-1)^{m-1}\Delta^{m-2}A_n = -\infty. \tag{3}$$

Since $(A_n)_{n\in Z}$ is bounded at ∞, (3) is a contradiction if $m = 2$. So, we suppose that $m > 2$ and we consider a positive constant γ_1. Then (3) implies the existence of an integer n_3 such that

$$(-1)^{m-1}\Delta^{m-2}A_n \leq -\gamma_1 \qquad \text{for every } n \geq n_3.$$

Thus, by applying the method used previously, we can obtain

$$\lim_{n\to\infty}(-1)^{m-1}\Delta^{m-3}A_n = -\infty.$$

Next, repeating the above procedure if $m > 3$, we finally find

$$\lim_{n\to\infty}(-1)^{m-1}A_n = -\infty,$$

which contradicts the fact that $(A_n)_{n\in Z}$ is positive and that this sequence is bounded at ∞. We have thus proved that (2) is true. If $m = 2$, the proof of the lemma is complete. If $m > 2$, then, repeating the above arguments, we can show that

$$(-1)^{m-2}\Delta^{m-2}A_n > 0 \qquad \text{for all } n \in Z.$$

By using this technique, we can complete the proof of the lemma.

Lemma 2. *Let m be odd. Then every positive solution of (E) is bounded at ∞.*

Proof. Assume, for the sake of contradiction, that (E) has a positive solution $(A_n)_{n\in Z}$ which is not bounded at ∞. Since m is odd, from (E) it follows that

$$\Delta^m A_n < 0 \qquad \text{for all } n \in Z. \tag{4}$$

Thus, we always have $m > 1$. Furthermore, (4) implies that, if $j \in \{1, \ldots, m-1\}$, then $\Delta^j A_n$ is either positive for all large n or negative for all large n.

In particular, since $(A_n)_{n\in Z}$ is not bounded at ∞, there exists an integer n_0 such that

(5) $$\Delta A_n > 0 \qquad \text{for every } n \geq n_0.$$

Furthermore, we have

(6) $$\Delta^{m-1} A_n > 0 \qquad \text{for all } n \in Z.$$

Indeed, in the opposite case we can apply the method used in the proof of Lemma 1 to obtain $\lim_{n\to\infty}(-1)^{m-1}A_n = -\infty$. So, since m is odd,

$$\lim_{n\to\infty} A_n = -\infty.$$

This contradicts the positiveness of $(A_n)_{n\in Z}$ and hence (6) is true. Now, we observe that, by (5), the sequence $(A_n)_{n\geq n_0}$ is strictly increasing. By using this fact, (6) and (E), for $n > N \equiv n_0 + l_0$ we obtain

$$\begin{aligned}
-(\Delta^{m-1}A_n)_{n=N} &< \Delta^{m-1}A_n - (\Delta^{m-1}A_n)_{n=N} \\
&= (\Delta^{m-1}A_n - \Delta^{m-1}A_{n-1}) + (\Delta^{m-1}A_{n-1} - \Delta^{m-1}A_{n-2}) + \cdots \\
&\qquad \cdots + [(\Delta^{m-1}A_n)_{n=N+1} - (\Delta^{m-1}A_n)_{n=N}] \\
&= \Delta^m A_{n-1} + \Delta^m A_{n-2} + \cdots + (\Delta^m A_n)_{n=N} \\
&= -\sum_{k=0}^{\infty} p_k A_{n-1-l_k} - \sum_{k=0}^{\infty} p_k A_{n-2-l_k} - \cdots - \sum_{k=0}^{\infty} p_k A_{N-l_k} \\
&< -p_0 A_{n-1-l_0} - p_0 A_{n-2-l_0} - \cdots - p_0 A_{N-l_0} \\
&\leq -p_0 A_{n_0}(n-N)
\end{aligned}$$

and so we arrive at the contradiction

$$n < N + \frac{1}{p_0 A_{n_0}}(\Delta^{m-1}A_n)_{n=N} \qquad \text{for every } n > N.$$

3. Proof of the theorem

In view of Lemma 2, for m odd there is no positive solution of (E) which is not bounded at ∞. Hence, it is enough to prove part (ii) of our theorem only for positive solutions which are bounded at ∞. Therefore, the proof of the theorem has been reduced to proving, for arbitrary m, the following result: Equation (E) has a positive solution which is bounded at ∞ if and only if $(*)$ has a root in $(0, 1)$.

Assume first that $(*)$ has a root $\lambda \in (0, 1)$. Then we set $A_n = \lambda^n$, $n \in Z$ and we obtain

$$\begin{aligned}
(-1)^{m+1}\Delta^m A_n + \sum_{k=0}^{\infty} p_k A_{n-l_k} &= (-1)^{m+1}(\lambda-1)^m \lambda^n + \sum_{k=0}^{\infty} p_k \lambda^{n-l_k} \\
&= [-(1-\lambda)^m + \sum_{k=0}^{\infty} p_k \lambda^{-l_k}]\lambda^n = 0
\end{aligned}$$

for all $n \in Z$. Thus, $(A_n)_{n\in Z}$ is a positive solution of (E) which obviously is bounded at ∞.

Suppose, conversely, that there is a positive solution $(A_n)_{n\in Z}$ of (E) which is bounded at ∞. Also assume, for the sake of contradiction, that the characteristic equation $(*)$ has no roots in $(0, 1)$. From Lemma 1 it follows that $\Delta A_n < 0$ for all $n \in Z$ and consequently $(A_n)_{n\in Z}$ is a strictly decreasing sequence. So, from (E) we obtain for every $n \in Z$

$$0 = (-1)^{m+1}\Delta^m A_n + \sum_{k=0}^{\infty} p_k A_{n-l_k} > (-1)^{m+1}\Delta^m A_n + \left(\sum_{k=0}^{\infty} p_k\right) A_n$$

and therefore

$$0 < \sum_{k=0}^{\infty} p_k < \infty. \tag{7}$$

Set

$$F(\lambda) = -(1-\lambda)^m + \sum_{k=0}^{\infty} p_k \lambda^{-l_k} \qquad \text{for } \lambda \in (0, 1].$$

Then

$$F(1) = \sum_{k=0}^{\infty} p_k \in (0, \infty).$$

Moreover, we have $F(\lambda) > -(1-\lambda)^m + p_1\lambda^{-l_1}$ for every $\lambda \in (0, 1)$, and so

$$F(0+0) = \infty.$$

Hence, as $F(\lambda) = 0$ has no roots in $(0, 1)$, there exists a positive number μ such that

$$-(1-\lambda)^m + \sum_{k=0}^{\infty} p_k \lambda^{-l_k} \geq \mu \qquad \text{for all } \lambda \in (0, 1). \tag{8}$$

Next, by taking into account (7), we put

$$\lambda_0 = 1 - \left(\sum_{k=0}^{\infty} p_k\right)^{1/m}, \qquad \text{and } \lambda_r = 1 - [(1-\lambda_{r-1})^m + \mu]^{1/m} \quad (r = 1, 2, \dots).$$

Furthermore, we define

$$A_n^{[0]} = A_n \qquad \text{for } n \in Z$$

and

$$A_n^{[r]} = \sum_{j=0}^{m-1} (1-\lambda_{r-1})^{m-1-j}(-1)^j \Delta^j A_n^{[r-1]} \qquad \text{for } n \in Z \quad (r = 1, 2, \dots).$$

Then, for any $r \in \{0, 1, \dots\}$, $(A_n^{[r]})_{n\in Z}$ is a positive solution of (E) which is bounded at ∞. Indeed, consider a positive solution $(\widetilde{A}_n)_{n\in Z}$ of (E) which is bounded at ∞. In view of Lemma 1, we have

$$(-1)^j \Delta^j \widetilde{A}_n > 0 \qquad \text{for all } n \in Z \quad (j = 0, 1, \dots, m-1, m).$$

Thus, if $j \in \{0, 1, \ldots, m-1\}$, then $((-1)^j \Delta^j \widetilde{A}_n)_{n\in Z}$ is a positive sequence which is strictly decreasing (and, therefore, bounded at ∞). Moreover, since (E) is linear and the coefficients p_k $(k = 0, 1, \ldots)$ and the indices l_k $(k = 0, 1, \ldots)$ are independent of n, it follows that, for each $j \in \{0, 1, \ldots, m-1\}$, the sequence $((-1)^j \Delta^j \widetilde{A}_n)_{n\in Z}$ is a solution of (E). Hence, each one of the sequences $((-1)^j \Delta^j \widetilde{A}_n)_{n\in Z}$ $(j = 0, 1, \ldots, m-1)$ is a positive solution of (E) which is bounded at ∞. Therefore, because of the linearity of (E), it follows that, if c_0, $c_1, \ldots, c_{m-1}$ are positive constants, then the sequence $(\sum_{j=0}^{m-1} c_j (-1)^j \Delta^j \widetilde{A}_n)_{n\in Z}$ is a positive solution of (E) which is bounded at ∞. Now, we can easily see that $1-\lambda_r > 0$ $(r = 0, 1, \ldots)$. So, by the above particular result and by mathematical induction, we can show that: If $r \in \{0, 1, \ldots\}$, then $(A_n^{[r]})_{n\in Z}$ is a positive solution of (E) which is bounded at ∞.

We have

(9)
$$(-1)^{m+1}\Delta^m A_n^{[r]} + (1-\lambda_r)^m A_n^{[r]} = A_{n+1}^{[r+1]} - \lambda_r A_n^{[r+1]} \qquad \text{for } n \in Z \quad (r = 0, 1, \ldots).$$

In fact, for any $r \in \{0, 1, \ldots\}$ and every $n \in Z$, we obtain

$$\begin{aligned}
A_{n+1}^{[r+1]} &- \lambda_r A_n^{[r+1]} \\
&= \sum_{j=0}^{m-1} (1-\lambda_r)^{m-1-j}(-1)^j \Delta^j A_{n+1}^{[r]} - \lambda_r \sum_{j=0}^{m-1} (1-\lambda_r)^{m-1-j}(-1)^j \Delta^j A_n^{[r]} \\
&= \sum_{j=0}^{m-1} (1-\lambda_r)^{m-1-j}(-1)^j (\Delta^j A_{n+1}^{[r]} - \Delta^j A_n^{[r]}) \\
&\qquad + (1-\lambda_r) \sum_{j=0}^{m-1} (1-\lambda_r)^{m-1-j}(-1)^j \Delta^j A_n^{[r]} \\
&= \sum_{j=0}^{m-1} (1-\lambda_r)^{m-1-j}(-1)^j \Delta^{j+1} A_n^{[r]} + \sum_{j=0}^{m-1} (1-\lambda_r)^{m-j}(-1)^j \Delta^j A_n^{[r]} \\
&= -\sum_{j=1}^{m} (1-\lambda_r)^{m-j}(-1)^j \Delta^j A_n^{[r]} + \sum_{j=0}^{m-1} (1-\lambda_r)^{m-j}(-1)^j \Delta^j A_n^{[r]} \\
&= (-1)^{m+1}\Delta^m A_n^{[r]} + (1-\lambda_r)^m A_n^{[r]}.
\end{aligned}$$

Now, we will prove that

$$(10) \qquad A_{n+1}^{[r+1]} - \lambda_r A_n^{[r+1]} < 0 \qquad \text{for all } n \in Z \quad (r = 0, 1, \ldots).$$

Indeed, in view of Lemma 1, the sequence $(A_n)_{n\in Z}$ is strictly decreasing. Therefore, we obtain for every $n \in Z$

$$(-1)^{m+1}\Delta^m A_n^{[0]} + (1-\lambda_0)^m A_n^{[0]} = (-1)^{m+1}\Delta^m A_n + \left(\sum_{k=0}^{\infty} p_k\right) A_n$$
$$< (-1)^{m+1}\Delta^m A_n + \sum_{k=0}^{\infty} p_k A_{n-l_k} = 0$$

and hence, by (9), we conclude that (10) is true for $r = 0$. Next, assuming that (10) holds for some $r \in \{0, 1, \dots\}$ we should prove that it is also true for $r+1$. By the inductive assumption,

$$A_{n+1}^{[r+1]} - \lambda_r A_n^{[r+1]} < 0 \qquad \text{for every } n \in Z.$$

This implies in particular that $\lambda_r > 0$. On the other hand, $\lambda_r < 1$. So, we must have $0 < \lambda_r < 1$. Furthermore, we have

$$A_n^{[r+1]} > \lambda_r^{-1} A_{n+1}^{[r+1]} \qquad \text{for } n \in Z.$$

By applying this inequality, we can verify that

$$A_{n-l_0}^{[r+1]} \geq \lambda_r^{-l_0} A_n^{[r+1]} \qquad \text{for all } n \in Z$$

and

$$A_{n-l_k}^{[r+1]} > \lambda_r^{-l_k} A_n^{[r+1]} \qquad \text{for all } n \in Z \quad (k = 1, 2, \dots).$$

Hence, from (E) we obtain for $n \in Z$

$$0 = (-1)^{m+1}\Delta^m A_n^{[r+1]} + \sum_{k=0}^{\infty} p_k A_{n-l_k}^{[r+1]}$$
$$> (-1)^{m+1}\Delta^m A_n^{[r+1]} + \left(\sum_{k=0}^{\infty} p_k \lambda_r^{-l_k}\right) A_n^{[r+1]}.$$

But (8) ensures that

$$\sum_{k=0}^{\infty} p_k \lambda_r^{-l_k} \geq (1-\lambda_r)^m + \mu = (1-\lambda_{r+1})^m.$$

So, in view of (9), we have for every $n \in Z$

$$0 > (-1)^{m+1}\Delta^m A_n^{[r+1]} + (1-\lambda_{r+1})^m A_n^{[r+1]}$$
$$= A_{n+1}^{[r+2]} - \lambda_{r+1} A_n^{[r+2]}.$$

That is, (10) is also satisfied for $r+1$.

Finally, since $A_n^{[r+1]} > 0$ for all $n \in Z$ $(r = 0, 1, \dots)$, from (10) it follows that

$$\lambda_r > 0 \qquad (r = 0, 1, \dots).$$

On the other hand, it is easy to verify that the sequence $(\lambda_r)_{r=0,1,\ldots}$ is strictly decreasing. So, $L \equiv \lim_{r\to\infty} \lambda_r$ exists and $0 \le L < \lambda_0 < 1$. Since

$$\lambda_r = 1 - [(1-\lambda_{r-1})^m + \mu]^{1/m} \qquad (r = 1, 2, \ldots),$$

we obtain

$$L = 1 - [(1-L)^m + \mu]^{1/m},$$

which gives $\mu = 0$, a contradiction. The proof of our theorem is complete.

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