PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 1, January 1990

THE DEATH OF AN INDEX THEOREM

MANUEL GONZALES AND ROBIN HARTE

(Communicated by Paul S. Muhly)

ABSTRACT. If the "index theorem" for Fredholm operators sometimes hold in the absence of an index [4], then also it sometimes fails in the presence of and index.

0. Call the bounded linear operator $T: X \to Y$ between normed spaces *weakly Fredholm* if it is both *essentially one-one*, in the sense ([3, Definition 6.4.1]) that

(0.1) $$T^{-1}(0) \text{ is finite dimensional},$$

and *essentially dense*, in the sense ([3, Definition 6.4.1]) that

$$Y/\,\mathrm{cl}(TX) \text{ is finite dimensional};$$

thus T is *Fredholm* if it is weakly Fredholm and *proper*, in the sense ([3, Definition 3.2.7]) that

(0.3) $$\mathrm{core}(T): X/T^{-1}(0) \to \mathrm{cl}(TX) \text{ is invertible},$$

where $\mathrm{core}(T)$ is induced by T in the obvious way. The product of weakly Fredholm operators is Fredholm; we have the implication

(0.4)
$$S, T \textit{ essentially one-one} \Rightarrow ST \textit{ essentially one-one} \Rightarrow T \textit{ essentially one-one}$$

since $T^{-1}(0) \subseteq (ST)^{-1}(0)$ and

(0.5) $$(ST)^{-1}(0)/T^{-1}(0) \cong T(X) \cap S^{-1}(0) \subseteq S^{-1}(0),$$

and the implication

(0.6) $$S, T \textit{ essentially dense} \Rightarrow ST \textit{ essentially dense} \Rightarrow S \textit{ essentially dense}$$

since

(0.7) $$T \textit{ essentially dense} \quad \Leftrightarrow \quad T^{\dagger} \textit{ essentially one-one},$$

Received by the editors March 10, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47B30.

where $T^\dagger: Y^\dagger \to X^\dagger$ is the *dual* or adjoint of $T: X \to Y$. If $T: X \to Y$ is weakly Fredholm then we may define its "index":

(0.8) $$\text{index}(T) = \dim T^{-1}(0) - \dim Y/\operatorname{cl}(TX),$$

where "dim" is the usual linear space *dimension.* Much if not all of the interest in the index lies in the fact that if $T: X \to Y$ and $S: Y \to Z$ are both Fredholm then ([3, Theorem 6.5.4];[5]; [6])

(0.9) $$\text{index}(ST) = \text{index}(S) + \text{index}(T);$$

indeed the essence of this extends [4] to pairs (S, T) for which S, T and ST all have "generalized inverses". When (0.9) holds we shall say that the pair (S, T) *has the index property.* Our first observation is that this does not hold universally:

1. **Example.** *If* $T = W: X \to Y = X$ *is one-one and dense and if* $S = I - f \odot e: X \to Z = X$ *with*

(1.1) $$f(e) = 1 \quad \textit{and} \quad e \notin W(X),$$

then (S, T) *does not have the index property.*

Proof. The rank one operator $f \odot e$ sends vectors $x \in X$ into $f(x)e$, and $S = S^2$ is a projection, which is Fredholm of index zero. The operator T is "weakly invertible", therefore also has index zero. Since T has dense range there is equality

(1.2) $$\operatorname{cl}(STX) = \operatorname{cl}(SX);$$

we claim that also

(1.3) $$(ST)^{-1}(0) = T^{-1}(0).$$

Indeed if $x \in X$ then $(ST)x = Tx - f(Tx)e$ and hence, since $e \notin T(X)$,

$$(ST)x = 0 \Rightarrow Tx = f(Tx)e \Rightarrow f(Tx) = 0 \Rightarrow Tx = 0.$$

Now $\operatorname{cl}(SX)$ has codimension 1 and $T^{-1}(0)$ has dimension 0, giving

$$\text{index}(ST) = 0 - 1 \neq 0 + 0 = \text{index}(S) + \text{index}(T). \quad \blacksquare$$

For a specific example take $X = l_2$ and $e_n = 1/n$ $(n \in \mathbb{N})$, $f(e) = 1$ and then define $(Wx)_n = e_n x_n$ $(n \in \mathbb{N})$ for each $x \in X$. Example 1 shows that the index of weakly Fredholm operators is not stable under finite rank perturbation, and not continuous with respect to the operator norm topology:

(1.4) $$\text{index}(T - fT \odot e) = \text{index}(ST) \neq \text{index}(T)$$

and

(1.5) $$\text{index}(T - tfT \odot e) = 0 \to 0 \neq \text{index}(ST) \quad \text{as } t \to 1.$$

The essence of Example 1 extends to more general products:

2. **Theorem.** *If* $T: X \to Y$ *is one-one and dense and if* $S: Y \to Z$ *is Fredholm of index zero, and satisfies*

$$T(X) \cap S^{-1}(0) = \{0\} \neq S^{-1}(0) \tag{2.1}$$

then (S, T) *does not have the index property. If in addition* $Z = X$ *and*

$$S(Y) \subseteq \mathrm{cl}(TSY) \tag{2.2}$$

then (ST, ST) *does not have the index property.*

Proof. Equality (1.2) holds exactly as in Example 1, and equality (1.3) follows from (2.1) and (0.5). If (2.2) is assumed then also

$$\mathrm{cl}(ST)^2(X) = \mathrm{cl}(SY) \quad \text{and} \quad (ST)^{-2}(0) = T^{-1}(0), \tag{2.3}$$

giving

$$\mathrm{index}(ST)^2 = \mathrm{index}(ST) \neq 2\,\mathrm{index}(ST). \quad \blacksquare \tag{2.4}$$

For a specific example take the direct sum of each of the operators in Example 1 with the identity $I: \mathbb{C} \to \mathbb{C}$. The discontinuity of the index is exhibited whenever the condition (2.1) of Theorem 2 is satisfied ([2, Theorem 1.1]), as well as instability with respect to finite rank perturbation.

We have already observed that if both factors S and T are actually Fredholm then (0.9) holds; the same is clear if either S or T is invertible:

$$S \text{ invertible} \quad \Rightarrow \quad \mathrm{index}(ST) = \mathrm{index}(T) \tag{2.5}$$

and

$$T \text{ invertible} \quad \Rightarrow \quad \mathrm{index}(ST) = \mathrm{index}(S). \tag{2.6}$$

We can extend this kind of derivation of (0.9):

3. **Theorem.** *If* $T: X \to Y$ *and* $S: Y \to Z$ *are weakly Fredholm and either*

$$T \text{ is onto} \tag{3.1}$$

or

$$S \text{ is bounded below} \tag{3.2}$$

or

$$S \text{ is one-one and } T \text{ is dense}, \tag{3.3}$$

then (S, T) *has the index property.*

Proof. If (3.3) holds then the isomorphism

$$T^{-1}(0) \times S^{-1}(0) \times Z/\mathrm{cl}(STX) \cong (ST)^{-1}(0) \times Y/\mathrm{cl}(TX) \times Z/\mathrm{cl}(SY) \tag{3.4}$$

is established in three stages: there is equality $(ST)^{-1}(0) = T^{-1}(0)$ since S is one-one, there is equality $Z/\mathrm{cl}(STX) = Z/\mathrm{cl}(SY)$ since T is dense, and finally $S^{-1}(0) = \{0\} = Y/\mathrm{cl}(TX)$. Counting dimensions on either side gives

(0.9). If we assume either that S is one-one or that T is onto then (0.5) holds with equality at the end, giving

(3.5) $$(ST)^{-1}(0)/T^{-1}(0) \cong S^{-1}(0)$$

and hence

(3.6) $$\dim(ST)^{-1}(0) = \dim S^{-1}(0) + \dim T^{-1}(0).$$

For equality

(3.7) $$\dim Z/\operatorname{cl}(STX) = \dim Z/\operatorname{cl}(SY) + \dim Y/\operatorname{cl}(TX)$$

we use the analogue of (3.5) with $(T^\dagger, S^\dagger)$ in place of (S, T): if either S is bounded below so that $S^\dagger$ is onto, or T is onto so that $T^\dagger$ is one-one, there is equality $S^\dagger(Z^\dagger) \cap T^{\dagger -1}(0) = T^{\dagger -1}(0)$. Taking stock, both (3.6) and (3.7) are established separately under each of the conditions (3.1), (3.2) and (3.3). ■

The second part of Theorem 2 shows that commutativity $ST = TS$ is not sufficient for the index property (0.9):

4. Theorem. *If* $T: X \to X$ *and* $S: X \to X$ *are weakly Fredholm and commute, and if either*

(4.1) $$T \text{ is weakly invertible}$$

or

(4.2) $$S \text{ is Fredholm of finite ascent and descent},$$

then (S, T) *has the index property.*

Proof. If T is weakly invertible, in the sense of being one-one and dense, then with no commutativity assumptions there is isomorphism

(4.3) $$T^{-1}(0) \times S^{-1}(0) \times X/\operatorname{cl}(STX) \cong (TS)^{-1}(0) \times X/\operatorname{cl}(TX) \times X/\operatorname{cl}(SX):$$

the derivation is very similar to that of (3.4). Specialising to $TS = ST$ now gives (3.4) and hence (0.9). If instead S is Fredholm of finite ascent and descent then ([1, Theorem 3.3]; [3, Theorem 7.3.6]) some power S^n has a commuting generalized inverse, and we can write

(4.4) $$S^n = UP = PU \qquad \text{with invertible } U \text{ and idempotent } P;$$

it now follows that P is in the "double commutant" of S^n, and in particular commutes with S, T and ST. We can therefore write

(4.5)
$$X = \begin{pmatrix} X_1 \\ X_0 \end{pmatrix}, \quad P = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} R_1 & 0 \\ 0 & R_0 \end{pmatrix} \quad \text{whenever } RS = SR,$$

where X_1 and X_0 are the range and null space of the projection P; in particular X_0 is finite dimensional, and S_1^n and hence also S_1 are invertible. Looking separately at $R(X)$ and $R^{-1}(0)$ shows that

(4.6) $$RP = PR \Rightarrow \operatorname{index}(R) = \operatorname{index}(R_1) + \operatorname{index}(R_0):$$

thus

$$(4.7)\qquad \begin{pmatrix} \mathrm{index}(T) \\ \mathrm{index}(S) \\ \mathrm{index}(ST) \end{pmatrix} = \begin{pmatrix} \mathrm{index}(T_1) \\ \mathrm{index}(S_1) \\ \mathrm{index}(S_1 T_1) \end{pmatrix} + \begin{pmatrix} \mathrm{index}(T_0) \\ \mathrm{index}(S_0) \\ \mathrm{index}(S_0 T_0) \end{pmatrix}.$$

Since X_0 is finite dimensional we must have ([3, Theorem 6.2.6]; [5])

$$(4.8)\qquad \mathrm{index}(T_0) = \mathrm{index}(S_0) = \mathrm{index}(S_0 T_0) = 0\,;$$

since S_1 is invertible (2.5) gives

$$(4.9)\qquad \mathrm{index}(S_1) = 0 \quad \text{and} \quad \mathrm{index}(S_1 T_1) = \mathrm{index}(T_1).$$

It follows that the third entry in each of the columns on the right hand side of (4.7) is the sum of the first two; this must also hold on the left hand side, giving (0.9). ■

Commutativity cannot be dropped from the assumptions of Theorem 4, since conditions (4.1) and (4.2) both hold in Example 1; in Theorem 2 the Fredholm operators of index zero are ([2]; [3, Theorem 6.5.2]) just the products of invertible operators and (Fredholm) projections. A variant of the second case of Theorem 4 holds whenever one factor is Fredholm: if $S = SS'S$ is Fredholm and T commutes with the projection $P = S'S$ then (0.9) holds (use the same decomposition of $X = Y$ as in the proof of Theorem 4), and similarly if instead $T = TT'T$ is Fredholm and S commutes with TT'. We conclude with an example which shows that the commutativity in (4.6) cannot be weakened to the invariance of the range of P under T:

5. **Theorem.** *If $A\colon X \to X$, $B\colon Y \to X$ and $D\colon Y \to Y$ satisfy*

$$(5.1)\qquad A \text{ is one-one and dense, and } D \text{ is dense}$$

and

$$(5.2)\qquad A(X) \cap B(Y) = \{0\} = B^{-1}(0) \cap D^{-1}(0)\,,$$

then

$$(5.3)\qquad \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} X \\ Y \end{pmatrix} \text{ is one-one and dense.}$$

Proof. If $x \in X$ and $y \in Y$ there is implication

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow Ax + By = 0 = Dy$$
$$\Rightarrow Ax = By = 0 = Dy \Rightarrow x = 0 = y\,;$$

if $f \in X^\dagger$ and $g \in Y^\dagger$ there is the implication

$$(f\ g) \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = (0\ 0) \Rightarrow fA = 0 = fB + gD \Rightarrow f = 0 = g. \quad ■$$

Here A is the restriction of $R = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ to the invariant subspace $\left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right) \subseteq \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$, and D the induced quotient mapping; if in particular A and D are both weakly Fredholm then so is R, and the extended version of (4.6) would be that $\mathrm{index}(R) = \mathrm{index}(A) + \mathrm{index}(D)$. Theorem 5 however says that $\mathrm{index}(R) = 0$ when the conditions (5.1) and (5.2) hold; to satisfy them without having $\mathrm{index}(D) = 0$ we can for example take $X = Y = l_2$ and $A = W$ as in Example 1, $D = V$ the backward shift (so $V(Y) = Y$ and $\dim V^{-1}(0) = 1$), and finally $B = f \odot e$ where $e \in X$ is not in the range $W(X)$ and $f \in Y^\dagger$ satisfies $f^{-1}(0) \cap V^{-1}(0) = 0$.

References

1. R. E. Harte, *Fredholm, Weyl and Browder theory*, Proc. Roy. Irish Acad. Sect. A **85** (1986), 151–176.
2. ——, *Regular boundary elements*, Proc. Amer. Math. Soc. **99** (1987), 328–330.
3. ——, *Invertibility and singularity*, Dekker, New York, 1988.
4. ——, *The ghost of an index theorem*, Proc. Amer. Math. Soc., **106** (1989), 1031–1034.
5. D. Sarason, *The multiplication theorem for Fredholm operators*, Amer. Math. Monthly **94** (1987), 68–70.
6. K. W. Yang, *Index of Fredholm operators*, Proc. Amer. Math. Soc. **41** (1973), 329–330.

Department of Mathematics, Universidad de Cantabria, 39005 Santander, Spain

Department of Mathematics, University College, Cork, Ireland

Current address (Robin Harte): Mathematical Sciences, University of Alaska, Fairbanks, Alaska 99775-1110