PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 2, February 1990

ON THE HARMONIC MAPS FROM $\mathbf{R}^2$ INTO H^2

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(Communicated by Barbara L. Keyfitz)

ABSTRACT. In this paper, we prove that normalized harmonic maps from $\mathbf{R}^2$ or $\mathbf{R}^2\backslash\{0\}$ into H^2 are just geodesics on H^2 and that the quasiconformal harmonic maps from $\mathbf{R}^2$ into H^2 are constant maps. We prove also that the only solution to $\Delta\alpha = \sinh\alpha$ on $\mathbf{R}^2\backslash\{0\}$ is the zero solution.

INTRODUCTION

Harmonic map is a common generalization of minimum submanifolds, harmonic functions and nonlinear σ-models ([3], [4]). We have a better understanding of harmonic maps from compact manifolds ([3], [5]), whereas about harmonic maps from noncompact manifolds we know rather little. Hence, it is very interesting to know harmonic from $\mathbf{R}^2$ into H^2 ([4]). Here, we give a description of harmonic maps from $\mathbf{R}^2$ into H^2 under some additional assumptions.

Theorem 1. *The only normalized harmonic map φ from $\mathbf{R}^2$ or $\mathbf{R}^2\backslash\{0\}$ into H^2 is geodesic, i.e., under suitable coordinate system, $\varphi(x,y) = \gamma(x)$, where $\gamma: \mathbf{R} \to H^2$ is a geodesic.*

The concept of normalized harmonic map is introduced in [7] and [8]. It is known that normalized harmonic maps from $\Omega \subset \mathbf{R}^2$ into H^2 exist locally ([7]). Our theorem shows however, that the global problem is quite different from the local one. (The definition of normalized harmonic map will be given in §2.)

The proof of the theorem is reduced to the discussion of the solution to the Sinh-Laplace equation:

$$\Delta\alpha = \sinh\alpha \qquad \text{on } \mathbf{R}^2 \text{ or } \mathbf{R}^2\backslash\{0\}.$$

The local version of the reduction is given by [7], and a global version will be given in §2.

Received by the editors October 4, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58E20.

We will prove the following theorem in §3:

Theorem 2. *Let* α *be a solution of*

$$\Delta\alpha = \sinh\alpha \qquad \text{on } \mathbf{R}^2\backslash\{0\}.$$

Then, $\alpha = 0$.

The solution $\alpha = 0$ is proved to correspond to geodesic on H^2, in §3.

Notice that normalized harmonic maps are nowhere conformal, hence, for the sake of completeness, we consider the harmonic maps which are not 'far' from the conformal ones. It is well known that there are no conformal harmonic maps from $\mathbf{R}^2$ into H^2 (except constant ones). We will prove that this is also true for quasi-conformal harmonic maps.

Definition. Let (M, g) and (N, h) be Riemannian manifolds, and $\varphi: M \to N$ a smooth map. φ is said to be quasi-conformal if there exists a constant K such that

$$h(\varphi)_* v, \varphi_* v) \leq K h(\varphi_* w, \varphi_* w),$$

$\forall p \in M, v, w \in T_p M, \|v\| = \|w\| = 1$.

This is a direct generalization of the conformal maps ($K = 1$ means conformal) and the quasi-conformal maps of ([2]). We will prove the following theorem in §4:

Theorem 3. *There exist no quasi-conformal harmonic maps from* $\mathbf{R}^2$ *into* H^2, *except constant maps.*

2. The reduction of the problem

Let $\varphi: \mathbf{R}^2 \to H^2$ be a smooth map. The energy of φ over a domain $U \subset \mathbf{R}^2$ is defined by:

$$E(\varphi, U) = \frac{1}{2}\int_U |d\varphi|^2\, dx\, dy.$$

φ is said to be harmonic if φ is a critical point of $E(\varphi, U)$ for all $U \subset \mathbf{R}^2$.

It is well known that H^2 can be realized as the 'unit sphere' in Minkowski space, i.e., $H^2 = \{x = (x_1, x_2, x_3) \in \mathbf{R}^{2,1}: x_1^2 + x_2^2 - x_3^2 = -1,\ x_3 \geq 1\}$. Now, $\varphi: \mathbf{R}^2 \to H^2$ can be reinterpreted as $\varphi: \mathbf{R}^2 \to \mathbf{R}^{2,1}$ valued in H^2. Then, the Euler-Lagrange equation for E is ([7, 8]):

$$\partial^2_{z\bar{z}}\varphi - (\partial_z\phi, \partial_{\bar{z}}\phi)\phi = 0, \qquad z = \tfrac{1}{2}(x + iy),$$

hence the norm (,) is defined by the metric of $\mathbf{R}^{2,1}$.

By this equation, we have conservation laws as $(\phi_z)^2_{\bar{z}} = (\phi_{\bar{z}})^2_z = 0$. If φ_z^2 has no zero, then there exists a holomorphic function $g(z)$ such that

$$g(z)^2 = \varphi_z^2.$$

Let $w = \frac{1}{2}(u + iv) = G(z) = \int_0^z g(z)$; then $(x, y) \to (u, v)$ is locally invertible and, under this transformation, we have $\phi_w^2 = \phi_{\overline{w}}^2 = 1$. It is easy to show that ϕ is harmonic if $\phi_w^2 = \phi_{\overline{w}}^2 = 1$ and $\phi_w, \phi_{\overline{w}}$ are linear independent. From the above discussion, it is natural to introduce the following definition.

Definition. The harmonic map $\varphi\colon \Omega \subset \mathbf{R}^2 \to H^2$ with $\phi_w^2 = \phi_{\overline{w}}^2 = 1$ is called a normalized harmonic map.

By Picard's theorem on the value distribution of holomorphic function, $G(z)$ takes values in $\mathbf{C}$ or $\mathbf{C}\backslash\{0\}$. Hence, we have

Theorem. *Any harmonic map from $\mathbf{R}^2$ into H^2 (without conformal point, i.e., $\phi_z^2 \neq 0$) can be decomposed as a conformal map into $\mathbf{R}^2$ or $\mathbf{R}^2\backslash\{0\}$ and a normalized harmonic map from $\mathbf{R}^2$ or $\mathbf{R}^2\backslash\{0\}$ into H^2 (in the latter case, we may have a multivalued map).*

Since $(\phi_w, \phi) = 0$ and $\phi_w^2 = 1$, we may assume that $\phi_u = ch\frac{1}{2}\alpha m$, $\phi_v = sh\frac{1}{2}\alpha n$ $l = \phi$ where l, m and n form an oriented orthonormal moving frame field for $\mathbf{R}^{2,1}$. It is easy to verify that:

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix}_u = \begin{pmatrix} & ch\frac{\alpha}{2} & \\ ch\frac{\alpha}{2} & & -\frac{1}{2}\alpha_v \\ & \frac{1}{2}\alpha_v & \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = U \begin{pmatrix} l \\ m \\ n \end{pmatrix},$$

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix}_v = \begin{pmatrix} & & sh\frac{\alpha}{2} \\ & & \frac{1}{2}\alpha_u \\ sh\frac{\alpha}{2} & -\frac{1}{2}\alpha_u & \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = V \begin{pmatrix} l \\ m \\ n \end{pmatrix}.$$

This system has solution only when $U_v - V_u + [U, V] = 0$, i.e.,

$$\Delta\alpha = \sinh\alpha \qquad \text{on } \mathbf{R}^2 \text{ or } \mathbf{R}^2\backslash\{0\}.$$

We restrict ourselves to single-valued normalized harmonic maps from $\mathbf{R}^2$ or $\mathbf{R}^2\backslash\{0\}$ into H^2. Then, for any such map (single-valued normalized harmonic map) from $\mathbf{R}^2$ or $\mathbf{R}^2\backslash\{0\}$ into H^2, there is a solution α of the Sinh-Laplace equation.

Lemma 1. *Let α be a solution of $\Delta\alpha = \sinh\alpha$ in $B_r(0)$, then $|\alpha(0)| \leq -4\ln r + \ln 17$. Here, $B_r(0)$ is the ball about 0 with radius $r \leq 1$.*

Proof. Let $\beta(\rho) = c_1 \ln(r^2 - \rho^2) + c_2$. We are going to find constants $c_1 < 0, c_2$, such that the following inequality holds:

$$\Delta\beta(\rho) \leq \sinh\beta(\rho).$$

Then, $\beta(\rho) \to +\infty$ as $\rho \to r$, hence

$$|\alpha(0)| \leq \beta(0) = 2c_1 \ln r + c_2,$$

by the standard maximum principle of ([6]).

In fact, if there exists a point $p \in B_r(0)$, such that $\alpha(p) > \beta(p)$, or $\alpha(p) < -\beta(p)$, assuming $\alpha(p) < -\beta(p)$, then $\alpha + \beta$ would have a negative minimum inside $B_r(0)$ (for example q), as $(\alpha+\beta)(p) \to +\infty$ when $p \to \partial B_r(0)$. But

$$\Delta(\alpha+\beta)(q) \le \sinh\alpha(q) + \sinh\beta(q) < 0$$

which is impossible. Similarly, we get a contradiction in the case of $\alpha(p) > \beta(p)$. Hence, $|\alpha(p)| < \beta(p)$ for $p \in B_r(0)$.

$$\begin{aligned}
\beta'(\rho) &= -2c_1 \frac{\rho}{r^2 - \rho^2}, \\
\beta''(\rho) &= -2c_1 \frac{(r^2+\rho^2)}{(r^2-\rho^2)^2}, \\
\Delta\beta(\rho) &= \beta''(\rho) + \frac{\beta'(\rho)}{\rho} \\
&= -4c_1 \frac{r^2}{(r^2-\rho^2)^2}, \\
\sinh\beta(\rho) &\ge (r^2-\rho^2)^{c_1}\sinh c_2, \qquad (c_1 < 0).
\end{aligned}$$

Let $c_1 = -2$, then $\Delta\beta \le \sinh\beta$ holds when $\sinh c_2 \ge 8r^2$. Hence, we can choose $c_2 = \ln 17$. Consequently,

$$|\alpha(0)| \le -4\ln r + \ln 17.$$

Lemma 2. *If $F\colon[0, +\infty) \to [0, +\infty)$ is a c^2 function such that $F(0) = 0$, $F'(t) \ge 0$ and $(F'(t)/t)' \ge cF^2(t)/t^3$ for some constant $c > 0$, then $F = 0$.*

Proof. Let $f(s) = F(\exp(s))$. Then $f'(s) = \exp(s)F'(\exp(s))$.

$$\begin{aligned}
f''(s) &= (e^{2s}F'(e^s)/e^s)' \\
&= 2e^{2s}(F'(e^s)/e^s) + e^{3s}(F'(t)/t)'|_{t=e^s} \\
&\ge cf^2(s).
\end{aligned}$$

Hence, we have $f'^2(s) - f'^2(s_0) \ge 2c/3(f^3(s) - f^3(s_0))$ for $s \ge s_0$. Assume $s_0 = \sup\{s\,; f(s) = 0\}$, then $f(s_0) = 0$ and $f(s) > 0$ for $s > s_0$. Hence, we have

$$\begin{gathered}
f'^2(s) \ge \tfrac{2}{3}cf^3(s), \\
f'(s)/f^{3/2}(s) \ge \sqrt{\tfrac{2}{3}c}, \\
\sqrt{\tfrac{2}{3}c}(s - s_1) \le \int_{s_1}^{s} \frac{f'(s)}{f^{3/2}}\,ds \le \int_{f(s_1)}^{+\infty} \frac{dx}{x^{3/2}} < \infty \qquad (s > s_1 > s_0).
\end{gathered}$$

Let $s \to +\infty$, we get a contradiction, if $F \ne 0$.

Theorem 2. *Let α be a solution of*

$$\Delta\alpha = \sinh\alpha$$

on $\mathbf{R}^2\backslash\{0\}$. Then, $\alpha = 0$.

Proof. Define $F(r) = \int_{B_r(0)} \alpha^2\, dx\, dy$. By Lemma 1, F is well defined.

$$\begin{aligned} F'(r) &= r\int_0^{2\pi} \alpha^2(r,\theta)\, d\theta \\ r\left(\frac{F'(\tau)}{r}\right)' &= 2r\int_0^{2\pi} \alpha_r\alpha\, d\theta \\ &= 2\int_{B_r} (|\nabla\alpha|^2 + \alpha\Delta\alpha) \qquad \text{(by Green's formula)} \\ &\geq c_1\int_{B_r} \alpha^4 \qquad (x\sinh x \geq cx^4) \\ &\geq c\left(\int_{B_r} \alpha^2\right)^2 / r^2 \qquad \text{(by Holder's inequality).} \end{aligned}$$

And by Lemma 2, $F = 0$. Hence $\alpha = 0$.

Theorem 1. *The normalized harmonic map φ from $\mathbf{R}^2$ or $\mathbf{R}^2\backslash\{0\}$ into H^2 is geodesic, i.e., under suitable a coordinate system, $\varphi(x,y) = \gamma(x)$, where $\gamma\colon \mathbf{R} \to H^2$ is a geodesic.*

Proof. First, we know from §2 that there exists a solution α of the Sinh-Laplace equation on $\mathbf{R}^2$ or $\mathbf{R}^2\backslash\{0\}$. By Theorem 2, $\alpha = 0$. Using the coordinate system of §2, we know that $\phi_v = 0$, $\phi_u^2 = 1$. Now, the equation of harmonic maps reduces to $\phi_{uu} - \phi = 0$. Hence $\phi(u,v) = \phi(u)$ is a geodesic on H^2.

4. Quasi-conformal harmonic maps

In this section, we will prove that there exist no quasi-conformal harmonic maps from $\mathbf{R}^2$ into negatively curved manifolds. First, let us prove a lemma.

Lemma 3. *Let $\phi\colon (M,g) \to (N,h)$ be a quasi-conformal map, then there exists a positive constant $c > 0$, such that*

$$g^{ij}g^{kl}\phi_i^\alpha\phi_j^\beta\phi_k^\gamma\phi_l^\delta(h_{\alpha\beta}h_{\delta\gamma} - h_{\alpha\delta}h_{\beta\gamma}) \geq c(h_{\alpha\beta}\phi_i^\alpha\phi_j^\beta g^{ij})^2$$

where $\phi_i^\alpha = \partial\phi^\alpha/\partial x_i$, etc.

Proof. For any $p \in M$ and $\varphi(p) \in N$, we choose orthonormal frames at p and $\varphi(p)$, $g_{ij}(p) = \delta_{ij}, h_{\alpha\beta}(\varphi(p)) = \delta_{\alpha\beta}$.

Since φ is quasi-conformal, we have

$$\begin{aligned} &\max_\theta(\phi_i^\alpha\phi_i^\alpha\cos^2\theta + 2\phi_i^\alpha\phi_j^\alpha\sin\theta\cos\theta + \phi_j^\alpha\phi_j^\alpha\sin^2\theta) \\ &\qquad \leq K\min_\theta(\phi_i^\alpha\phi_i^\alpha\cos\theta + 2\phi_i^\alpha\phi_j^\alpha\sin\theta\cos\theta + \phi_j^\alpha\phi_j^\alpha\sin^2\theta). \end{aligned}$$

(where we take summation only with respect to α). It is easy to prove that

$$\sum_\alpha \phi_i^{\alpha 2} + \sum_\alpha \phi_j^{\alpha 2} + \sqrt{4\left(\sum_\alpha \phi_i^\alpha \phi_j^\alpha\right)^2 + \left(\sum_\alpha \phi_i^{\alpha 2} - \sum_\alpha \phi_j^{\alpha 2}\right)^2}$$
$$\leq K\left(\sum_\alpha \phi_i^{\alpha 2} + \sum_\alpha \phi_j^{\alpha 2} - \sqrt{4\left(\sum_\alpha \phi_i^\alpha \phi_j^\alpha\right)^2 + \left(\sum_\alpha \phi_i^{\alpha 2} - \sum_\alpha \phi_j^{\alpha}\right)^2}\right).$$

Hence we have

$$\sum_\alpha \phi_i^{\alpha 2} \sum_\alpha \phi_j^{\alpha 2} - \left(\sum_\alpha \phi_i^\alpha \phi_j^\alpha\right)^2 \geq c\left(\sum_\alpha \phi_i^{\alpha 2} + \sum_\alpha \phi_j^{\alpha 2}\right)^2.$$

Taking summation over all (i, j) and using Holder's inequality, we have

$$g^{ij} g^{kl} \phi_i^\alpha \phi_j^\beta \phi_k^\gamma \phi_l^\delta (h_{\alpha\beta} h_{\delta\gamma} - h_{\alpha\delta} h_{\beta\gamma}) \geq c(h_{\alpha\beta} \phi_i^\alpha \phi_j^\beta g^{ij})^2.$$

Theorem 3. *There exists no quasi-conformal harmonic map* $\varphi: \mathbf{R}^2 \to N$ (*except constant maps*) *if* $Riem^N \leq -a^2 < 0$.

Proof. Let $e(\varphi) = h_{\alpha\beta}\phi_i^\alpha \phi_j^\beta g^{ij}$, and $F(r) = \int_{B_r(0)} e(\varphi)\, dx\, dy$. Then

$$F'(r) = r\int_0^{2\pi} e(\varphi)(r, \theta)\, d\theta,$$

$$\begin{aligned}
r(F'(r)/r)' &= 2r\int_{S_r} h_{\alpha\beta}\phi_i^\alpha \nabla_{\partial/\partial r}(\phi^{\beta i})\, d\theta \\
&= 2\int_{B_r} h_{\alpha\beta}\phi_{i,j}^\alpha \phi^{\beta i,j} + 2\int_{B_r} h\alpha\beta\phi_i^\alpha \phi_j^{\beta i,j} \\
&\geq 2\int_{B_r} h_{\alpha\beta}\phi_i^\alpha \phi_j^{\beta i,j} \\
&\geq 2\int_{B_r} h_{\alpha\beta}\phi^{\alpha i}\phi_{ji}^{\beta j} \\
&\qquad - 2\int_{B_r}\sum_{i,j} R^N\left(\phi_*\left(\frac{\partial}{\partial x^i}\right), \phi_*\left(\frac{\partial}{\partial x^j}\right), \phi_*\left(\frac{\partial}{\partial x^i}\right), \phi_*\left(\frac{\partial}{\partial x^j}\right)\right) \\
&\geq 2a^2\int_{B_r} \phi_i^\alpha \phi_j^\beta \phi^{\gamma i}\phi^{\delta j}(h_{\alpha\beta}h_{\delta\gamma} - h_{\alpha\delta}h_{\beta\gamma}) \\
&\geq 2a^2 c\int_{B_r} e^2(\varphi) \qquad \text{(by our Lemma 3)} \\
&\geq cF^2(r)/r^2 \qquad \text{(by Holder's inequality)}.
\end{aligned}$$

By Lemma 2, $F = 0$. Hence φ is a constant map.

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