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SPLITTING THEOREM FOR HOMOLOGY OF GL(R)

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ABSTRACT. It is proved that if $\{M_n\}$ is a stable system of coefficients for $\mathrm{Gl}_n(R)$, and $H_0(\mathrm{Gl}(R), \lim(M_n))$ contains $\mathbf{Z}$, then for any j, the group $H_j(\mathrm{Gl}(R), \lim(M_n))$ contains $H_j(\mathrm{Gl}(R), Z)$ as a direct summand. Now let $\mathrm{Gl}(\mathbf{Z})$ act on $M(\mathbf{Z})$ (matrices over $\mathbf{Z}$) by conjugation. Then our theorem implies that the trace map $\mathrm{tr}\colon M(\mathbf{Z}) \to \mathbf{Z}$ is a split epimorphism on homology.

0. INTRODUCTION

In the late seventies the development of algebraic K-theory forced mathematicians to study the following question: does the inclusion of $\mathrm{Gl}_n(R)$ into $\mathrm{Gl}_{n+1}(R)$ induce the isomorphism on homology with trivial or twisted coefficient systems? (R is any ring). The most general answer to this question was given by W. van der Kallen in [6]. He solved that problem in affirmative for rings which satisfy Bass's stable range condition (see [1] for the definition). In [2] we used the homological stability theorems to get information about the homology of $\mathrm{Gl}_n(R)$. More precisely, we gave there a description of coefficient systems for $\mathrm{Gl}_n(R)$ which give stably trivial homology. In the following note we study the opposite situation and describe systems of coefficients which give nontrivial homology of $\mathrm{Gl}(R) = \varinjlim_n \mathrm{Gl}_n(R)$.

§1

1.1. **Definition.** Let R be any ring. Let M_n be a $\mathrm{Gl}_n(R)$-module for any natural n. Assume that we have homomorphisms $F_n\colon M_n \to M_{n+1}$ which are equivariant under the inclusion of $\mathrm{Gl}_n(R)$ into $\mathrm{Gl}_{n+1}(R)$ given by $B \to \begin{vmatrix} B0 \\ 01 \end{vmatrix}$. Then we say that the set $\{M_n\}$ forms a system of coefficients for $\mathrm{Gl}_n(R)$.

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In this section we will study only systems $\{M_n\}$ of coefficients for $\mathrm{Gl}_n(R)$ which satisfy the following conditions:

(a) $\{M_n\}$ is a stable system of coefficients for $\mathrm{Gl}_n(R)$ (i.e. for any i and sufficiently large n the map $H_i(\mathrm{Gl}_n(R); M_n) \to H_i(\mathrm{Gl}_{n+1}(R); M_{n+1})$ is an isomorphism);

(b) for any n and k, the image of M_n in M_{n+k} is fixed pointwise by the subgroup $\begin{vmatrix} I_n & 0 \\ 0 & \mathrm{Gl}_k \end{vmatrix}$ of $\mathrm{Gl}_{n+k}(R)$.

For convenience we will assume that for any n, the module M_n is finitely generated as an abelian group, and R always denotes a ring with unit which satisfies Bass's stable range condition. We will denote $\varinjlim_n \mathrm{Gl}_n(R)$ as $\mathrm{Gl}\,R$ and $\varinjlim_n M_n$ as M.

1.2. **Theorem.** *Assume that* $\varinjlim_n H_0(\mathrm{Gl}_n(R); M_n) = A \neq 0$. *Let* $f\colon \mathbf{Z} \to A$ *be a homomorphism and* $f_*\colon H_j(\mathrm{Gl}\,R; \mathbf{Z}) \to H_j(\mathrm{Gl}\,R; A)$ *the induced homomorphism. Then, for any* $j \geq 0$, f_* *can be factored through* $H_j(\mathrm{Gl}\,R; M)$. *In particular, if* f_* *is nontrivial, then* $H_j(\mathrm{Gl}\,R; M) \neq 0$.

Proof. Let n be such a large natural number that $H_j(\mathrm{Gl}_n(R); M_n) = H_j(\mathrm{Gl}\,R; M)$ and $H_j(\mathrm{Gl}_n(R); \mathbf{Z}) = H_j(\mathrm{Gl}\,R; \mathbf{Z})$. Then by [5] $H_0(\mathrm{Gl}_n(R); M_n) = A$ and the quotient map $M_n \xrightarrow[j_n]{} A$ is a $\mathrm{Gl}_n(R)$-homomorphism where $\mathrm{Gl}_n(R)$ acts trivially on A. We have the following commutative diagram of $\mathrm{Gl}_n(R)$-modules:

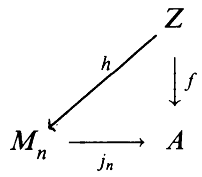
$$\begin{array}{ccccc} M_n & \xrightarrow{F_n} & \cdots & \xrightarrow{F_{2n-1}} & M_{2n} \\ \downarrow{\scriptstyle j_n} & & & & \downarrow{\scriptstyle j_{2n}} \\ A & = & \cdots & = & A \end{array}$$

We have the following commutative diagram of groups and modules:

$$(*)\qquad \begin{array}{ccc} & & (\mathrm{Gl}_n(R), \mathbf{Z}) \\ & \swarrow{\scriptstyle G} & \downarrow{\scriptstyle F} \\ & & (\mathrm{Gl}_{2n}(R), M_{2n}) \\ & & \downarrow{\scriptstyle (\mathrm{id}, j_{2n})} \\ (\mathrm{Gl}_{2n}(R), \mathbf{Z}) & \xrightarrow{(\mathrm{id}, f)} & (\mathrm{Gl}_{2n}(R), A) \end{array}$$

where G is defined as the map which takes a matrix $B \in \mathrm{Gl}_n(R)$ to $\begin{vmatrix} I_n & 0 \\ 0 & B \end{vmatrix}$ and is equal to identity on coefficients. To define F we use condition (b) about

$\{M_n\}$. Let h be a map in the category of abelian groups given by the following commutative diagram:



We can then define F as the same map as G on the groups and as $F_{2n-1} \circ \cdots \circ F_n \circ h$ on coefficients. By condition (b), F is well defined.

Looking now at diagram $(*)$ after taking the jth homology group, we see that the map $((\mathrm{id}, f) \circ G)_*\colon H_j(\mathrm{Gl}_n(R); \mathbf{Z}) \to H_j(\mathrm{Gl}_{2n}(R); A)$ can be factored through the group $H_j(\mathrm{Gl}_{2n}(R); M_{2n})$. On the other hand, it is obvious that $((\mathrm{id}, f) \circ G)_*$ agrees with $f_*\colon H_j(\mathrm{Gl}\, R; \mathbf{Z}) \to H_j(\mathrm{Gl}\, R; A)$— we have only to use the fact that the lower and upper inclusions of $\mathrm{Gl}_n(R)$ into $\mathrm{Gl}_{2n}(R)$ give the same map on homology with trivial coefficients. Thus f_* factors through $H_j(\mathrm{Gl}_{2n}(R); M_{2n})$, and by the assumption on n this last group is isomorphic to $H_j(\mathrm{Gl}\, R; M)$.

1.3. *Remark.* The following two conditions are equivalent for any group G and any G-module M which is finitely generated as an abelian group:

(1) $H_0(G; M) \neq 0$;
(2) there exists a G-epimorphism f, from M to the trivial G-module $\mathbf{Z}$ or $\mathbf{Z}_p$.

Proof. If $H_0(G; M) \neq 0$, take as f the natural projection $M \to H_0(G; M)$ composed with a projection from $H_0(G; M)$ onto any of its factors.

If there exists f as in condition (2), then for any $g \in G$ and any $m \in M$, $f(gm) = gf(m)$; so $f((g-1)m) = 0$, and hence f induces an epimorphism $H_0(G; M) \to \mathbf{Z}$ or $\mathbf{Z}_p$.

1.4. **Corollary.** *Let R be any commutative ring with unit satisfying the hypothesis of Theorem* 1.2. *Let M_n denotes the $n \times n$ matrices with entries in R on which the group $\mathrm{Gl}_n(R)$ acts by conjugation. Then the the trace map $tr\colon M \to R$ induces a nontrivial map on homology provided that $H_*(\mathrm{Gl}\, R; \mathbf{Z}) \to H_*(\mathrm{Gl}\, R; R)$ is nontrivial* (*this map is induced by the obvious homomorphism $\mathbf{Z} \to R$*).

1.5. *Remark.* Now let R denote any ring with unit satisfying Bass's stable range conditions and let P denote the R-bimodule. Let $H_*^{\mathrm{Hoch}}(R, P)$ denote the Hochschild homology of R with coefficients in P. Let $M(P)$ denote the full group of matrices over P on which $\mathrm{Gl}\, R$ acts by conjugation. It was shown by Kassel in [7] that

$$H_0(\mathrm{Gl}\, R, M(P)) = H_0^{\mathrm{Hoch}}(R, P)$$

and

$$H_1(\mathrm{Gl}\, R, M(P)) = H_1(\mathrm{Gl}\, R, \mathbf{Z}) \otimes H_0^{\mathrm{Hoch}}(R, P) \oplus H_1^{\mathrm{Hoch}}(R, P).$$

Our Theorem 1.2 implies that $H_j(\mathrm{Gl}\,R, \mathbf{Z}) \otimes H_0^{\mathrm{Hoch}}(R, P)$ is a direct summand of $H_j(\mathrm{Gl}\,R, M(P))$ provided that $H_0^{\mathrm{Hoch}}(R, P)$ is $\mathbf{Z}$-free (see also [5]).

1.6. **Corollary.** *Now let $R = \mathbf{Z}$ and M be as in* 1.4. *Then $tr\colon M \to \mathbf{Z}$ gives a split epimorphism on homology.*

It is known that in 1.6, tr_* is an isomorphism when we tensor homology groups with Q ([4]); but such splitting as in 1.6 is obvious. Using Kassel's results from [7], it is easy to calculate that tr_* from 1.6 is an isomorphism on H_0 and H_1.

1.7. *Remark.* Condition (b) is not very restrictive. The main examples of stable systems of coefficients for $\mathrm{Gl}_n(R)$ are obtained by applying any functor of finite degree to R^n considered in an obvious way as a $\mathrm{Gl}_n(R)$ module (see [3] or [6]). But it is easy to see that in such a way we can obtain only systems which satisfy (b).

1.8. *Remark.* Theorem 1.2 remains true when we replace the groups $\mathrm{Gl}_n R$ by any family of subgroups $G_n \subseteq \mathrm{Gl}_n R$ such that

(1) the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Gl}_n R & \longrightarrow & \mathrm{Gl}_{n+1} R \text{ (the standard "upper" inclusion)} \\ \cup & & \cup \\ G_n & \longrightarrow & G_{n+1} \end{array}$$

(2) the "lower" and "upper" inclusions $G_n \to G_{2n}$ are defined and conjugated, and they agree with the "lower" and "upper" inclusions of $\mathrm{Gl}_n R$ into $\mathrm{Gl}_{2n} R$ for some sufficiently large n (depending on j).

In particular, it is true for $\mathrm{Sl}_n R$ in a commutative case and more generally for $E_n(R)$.

§2

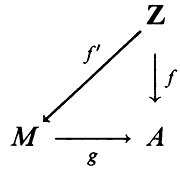
In this section we drop the assumption that our system of coefficients is stable. We call $\varinjlim_n H_0(\mathrm{Gl}_n(R); M_n)$ as A.

2.1. **Theorem.** *Assume that there is a $\mathbf{Z}$-homomorphism $f\colon \mathbf{Z} \to A$ which gives us a nontrivial map*

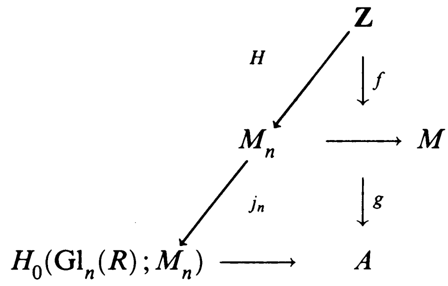
$$f_*\colon \varinjlim_n H_j(\mathrm{Gl}_n(R); \mathbf{Z}) \to \varinjlim_n H_j(\mathrm{Gl}_n(R); A).$$

Then $\varinjlim_n H_j(\mathrm{Gl}_n(R); M_n)$ is nontrivial.

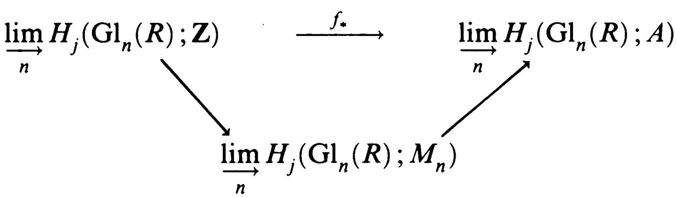
Proof. Let $M = \varinjlim_n M_n$. There is a natural epimorphism $g: M \to A$. Let f' be a map obtained from the following diagram



For sufficiently large n we can easily find a map H which fits into the following commutative diagram:



We can now apply precisely the same method as in the proof of 1.2, using our new map H instead of h, replacing j_{2n} by j_{2n} composed with the standard homomorphism $H_0(\mathrm{Gl}_{2n} R; M_{2n}) \to A$, and taking sufficiently large n. We then obtain the following commutative diagram:



The existence of such a diagram immediately implies our Theorem 2.1.

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