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ON ASYMPTOTIC BEHAVIOR OF THE MASS OF RAYS

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ABSTRACT. We consider the measure of the set of all unit vectors tangent to rays emanating from a point p in a finitely connected complete open Riemannian 2-manifold M. If M with one end admits total curvature $c(M)$, then this measure tends to $\min\{2\pi\chi(M)-c(M), 2\pi\}$ as p tends to infinity, where $\chi(M)$ is the Euler characteristic.

0. INTRODUCTION

Let M be a complete, noncompact, connected, oriented and finitely connected Riemannian 2-manifold without boundary. The total curvature $c(M)$ of M is defined by the improper integral over M of Gaussian curvature G:

$$c(M) := \int_M G dM,$$

where dM is the area element of M. It is a well-known theorem due to Cohn-Vossen [1] that if M admits total curvature, then $c(M) \leq 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. We study asymptotic behavior of the mass of rays in terms of the total curvature of a complete open surface.

A ray $\gamma: [0, \infty) \to M$ is defined to be a geodesic any subarc of which is a minimizing segment joining its endpoints. We denote the tangent space of M at p by M_p. Let $S_p \subset M_p$ be the set of all unit vectors at p and let $A_p \subset S_p$ be the set of all unit vectors tangent to rays emanating from p. We denote by "meas" the Lebesgue measure on the unit circle S_p with the total measure 2π. Since the limit of a sequence of rays in M is a ray, A_p is a closed and measurable subset of S_p and the function $p \mapsto \text{meas}(A_p)$ is upper-semicontinous. Thus this function is locally integrable in the sense of Lebesgue.

The first result on the relation between the total curvature and the measure of A_p was obtained by Maeda.

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Theorem (Maeda [6]). *If M is homeomorphic to R^2 and if it has nonnegative Gaussian curvature everywhere, then*

$$\inf_{p\in M} \operatorname{meas}(A_p) = 2\pi - c(M).$$

In [10], Shiga extended this result to the case when the sign of Gaussian curvature changes. Moreover, Oguchi [8] extended their results to the case when M has only one end. Shiohama proved the following integral formula for the mass of rays.

Theorem (Shiohama [13]). *Assume that M with one end admits total curvature with $2\pi\chi(M) - c(M) < 2\pi$. If $\{K_j\}$ is a monotone increasing sequence of compact sets with $\bigcup K_j = M$, then*

$$(*) \qquad \lim_{j\to\infty} \frac{\int_{K_j} \operatorname{meas}(A_p)dM}{\int_{K_j} dM} = 2\pi\chi(M) - c(M).$$

The following Theorem A which will be proved in §2 plays an essential role throughout this paper.

Theorem A. *Assume that M with one end admits total curvature. Let $\{p_j\}$ be an arbitrary divergent sequence of points of M. Then,*

$$\lim_{j\to\infty} \operatorname{meas}(A_{p_j}) = \min\{2\pi\chi(M) - c(M), 2\pi\}.$$

In the case when $2\pi\chi(M)-c(M) < 2\pi$, Theorem A was proved by Shiohama in the proof of the integral formula $(*)$. A crucial point of the proof of $(*)$ in the case when $2\pi\chi(M) - c(M) < 2\pi$ is nonexistence of straight lines. We emphasize that in our case M admits straight lines and this situation makes the proof difficult. To overcome this difficulty we need delicate arguments as developed in Lemmas 2.1, 2.2, 2.3, and 2.4. In §2, we will prove Theorem A in the case when $2\pi\chi(M) - c(M) \geq 2\pi$. We can extend Theorem A to the case when M has more than one end as stated in Theorem B.

In §3, we will discuss the case when M has finitely many ends. To state Theorem B some definitions and notations are needed. Assume that M is finitely connected with k ends and that M admits total curvature. Let K be a compact domain on M such that $M - \operatorname{Int}(K)$ is a union of disjoint closed half cylinders $U_1, \dots, U_k$ (we call them tubes) and ∂K consists of k simple closed piecewise smooth curves. For any domain D bounded by piecewise smooth curves $c_1, \dots, c_n$ each of which is parametrized positively by arc length relative to D, we denote by $\kappa(D)$ the sum of curvature integrals of $c_1, \dots, c_n$ and of the outer angles at all the vertices of D. If we set $s_i(M) := -c(U_i) - \kappa(U_i)$ for $i = 1, \dots, k$, then

$$\sum_{1\leq i\leq k} s_i(M) = 2\pi\chi(M) - c(M).$$

The value $s_i(M)$ does not depend on the choice of tube U_i by the Gauss-Bonnet Theorem.

With these notations Theorem B is stated as follows.

Theorem B. *Assume that M with k ends admits total curvature. Let U_i be a tube of M and let $\{p_j\}$ be an arbitrary divergent sequence of points of U_i. Then*

$$\lim_{j\to\infty} \operatorname{meas}(A_{p_j}) = \min\{s_i(M), 2\pi\}.$$

The following Corollaries C, D, and Theorem E are straightforward consequences of Theorem B and the isoperimetric inequality stated in Lemma 1.3 (2) (see [14]). The proofs are omitted here.

Corollary C. *Under the same assumption as in Theorem B, let $\{p_j\}$ be an arbitrary divergent sequence of points of M. Then,*

$$\begin{aligned}\min_{1\le i\le k}\{s_i(M), 2\pi\} &\le \liminf_{j\to\infty} \operatorname{meas}(A_{p_j})\\ &\le \limsup_{j\to\infty} \operatorname{meas}(A_{p_j})\\ &\le \max_{1\le i\le k} \min\{s_i(M), 2\pi\}.\end{aligned}$$

Theorem D. *Assume that M with k ends admits total curvature. Let $\{K_j\}$ be a monotone increasing sequence of compact subsets of M with $\bigcup K_j = M$. Then we have*

$$\begin{aligned}\min_{1\le i\le k}\{s_i(M), 2\pi\} &\le \liminf_{j\to\infty} \frac{\int_{K_j} \operatorname{meas}(A_p)dM}{\int_{K_j} dM}\\ &\le \limsup_{j\to\infty} \frac{\int_{K_j} \operatorname{meas}(A_p)dM}{\int_{K_j} dM}\\ &\le \max_{1\le i\le k} \min\{s_i(M), 2\pi\}.\end{aligned}$$

Theorem E. *Assume that M with k ends admits total curvature. Let c be a simple closed smooth curve in M and let $B(t) := \{x \in M; d(x, c) < t\}$. Then we have*

$$\lim_{t\to\infty} \frac{\int_{B(t)} \operatorname{meas}(A_p)dM}{\int_{B(t)} dM} = \begin{cases} \frac{\sum_{1\le i\le k} s_i(M)\min\{s_i(M), 2\pi\}}{2\pi\chi(M) - c(M)} & \text{if } 2\pi\chi(M) - c(M) > 0,\\ 0 & \text{if } 2\pi\chi(M) - c(M) = 0.\end{cases}$$

1. Preliminaries

In this section, we state the notations and lemmas used for the proof of our results. Let M be a finitely connected complete open 2-manifold admitting total curvature. Let $D \subset M$ be a domain as stated in §0. Then $\kappa(D)$ has the following properties.

(1.1) $\kappa(D) = -\kappa(M - D)$.

(1.2) If D is bounded, then $c(D) = 2\pi\chi(D) - \kappa(D)$.

(1.3) Assume that ∂D consists of a curve c homeomorphic to a line such that $c|(-\infty, a], c|[b, \infty)$ are geodesics for some $a, b \in R$. Then,

$$c(D) \le 2\pi\chi(D) - \pi - \kappa(D).$$

(1.4) In (1.3), if $d_D(c(t), c(-t)) \geq 2t - r$ for all $t \geq 0$ and for some constant $r \geq 0$, then

$$c(D) \leq 2\pi\chi(D) - 2\pi - \kappa(D)$$

where d_D is the inner distance on $\mathrm{Cl}(D)$, the closure of D, induced from the Riemannian structure of M.

(1.1) is obvious. (1.2) follows from the Gauss-Bonnet Theorem. (1.3) and (1.4) are due to Cohn-Vossen [2].

For the rest of this section we assume that M has only one end. The following Lemma 1.1 plays an important role for the proof of Lemmas 2.1 and 2.2.

Lemma 1.1 (Shiga [10]). *Let σ, γ be rays emanating from a point p in M. Assume that $\sigma \cup \gamma$ bounds a domain D and that $\mathrm{Int}(D)$ does not contain any ray emanating from p. If θ is the inner angle of D at p, then we have*

$$c(D) = 2\pi\chi(D) - 2\pi + \theta.$$

The proof of Lemma 1.1 proceeds in outline as follows. Since $\mathrm{Int}(D)$ does not contain any ray emanating from p, there is a divergent sequence $\{q_j\}$ in D with the property that there exist two minimizing geodesics ξ_j and η_j joining p to q_j in D such that $\lim \xi_j = \gamma$ and $\lim \eta_j = \sigma$ and such that the inner angle at q_j of the disk domain D_j in D bounded by ξ_j and η_j tends to zero as $j \to \infty$. The sequence $\{D_j\}$ of disk domains is monotone increasing and satisfies $\bigcup D_j = D$.

For a point $p \in M$ and for $u \in A_p$ let $\gamma_u(t) := \exp_p tu$ for $t \geq 0$. Let K be an arbitrary fixed compact domain bounded by a piecewise smooth closed curve such that $M - K$ is an open half cylinder. For any geodesic γ passing through a point of K, set

$$t_0(\gamma) := \min\{t; \gamma(t) \cap \partial K\}, \qquad t_1(\gamma) := \max\{t; \gamma(t) \cap \partial K\}.$$

For a point $p \in M - K$, set

$$A_p(K) := \{v \in A_p; \gamma_v([0, \infty)) \cap K \neq \emptyset\},$$
$$A'_p(K) := \{v \in A_p; \gamma_v([0, \infty)) \cap \mathrm{Int}(K) = \emptyset\}.$$

For $u, v \in A_p(K)$, the two subarcs $\gamma_u([0, t_0(\gamma_u)])$, $\gamma_v([0, t_0(\gamma_v)])$ of the rays γ_u, γ_v and a subarc of ∂K joining $\gamma_u(t_0(\gamma_u))$ to $\gamma_v(t_0(\gamma_v))$ together form a simple closed curve in $M - \mathrm{Int}(K)$ which bounds an open disk domain $\Delta_p(u, v)$ in $M - K$. If we set

$$\Delta_p(K) := \bigcup_{u, v \in A_p(K)} \Delta_p(u, v)$$

and if $\theta_p(K)$ is the inner angle of $\Delta_p(K)$ at p, then the following lemma is true.

Lemma 1.2 ([14]). *For any divergent sequence $\{p_i\}$ of points in $M - K$, $\theta_{p_i}(K)$ tends to zero as $i \to \infty$.*

The following isoperimetric inequality is used for the proof of our integral formulas. For a point $p \in M$ let

$$S_t(p) := \{x \in M ; d(p, x) = t\}, \qquad B_t(p) := \{x \in M ; d(p, x) < t\}.$$

Lemma 1.3 (Hartman [4] and Shiohama [11]).

(1) *There exists a constant $R > 0$ such that for almost all $t \geq R$, $S_t(p)$ is a simple closed curve of class C^∞ except finitely many cut points from p.*

(2) *We have*

$$\lim_{t\to\infty} \frac{L(S_t(p))}{t} = \lim_{t\to\infty} \frac{2\,\mathrm{Area}(B_t(p))}{t^2} = \lim_{t\to\infty} \frac{L(S_t(p))^2}{2\,\mathrm{Area}(B_t(p))} = 2\pi\chi(M) - c(M),$$

where $L(\alpha)$ is the length of a curve α.

2. The case when M has one end

In this section, *we assume that M has only one end and that M admits total curvature with $2\pi\chi(M) - c(M) \geq 2\pi$.* Let K be a compact domain such that $M - K$ is an open half cylinder and p a point in $M - K$. Assume that $A'_p(K)$ is nonempty. Then we denote by $D_p(K)$ the unique component of $M - \{\exp_p tv ; v \in A'_p(K),\ t \geq 0\}$ such that $K \subset D_p(K)$.

Lemma 2.1. *Let $\{p_i\}$ be a divergent sequence of points of M. Assume that for any compact set K such that $M - K$ is an open half cylinder, $A'_{p_i}(K)$ is nonempty for all sufficiently large i and that the inner angle θ_i of $D_{p_i}(K)$ at p_i tends to zero as $i \to \infty$. Then*

$$\lim_{i\to\infty} \mathrm{meas}(A_{p_i}) = 2\pi.$$

Proof. Let ε be an arbitrary given positive number. Let K be a compact subset of M such that $M - K$ is an open half cylinder and such that

$$\int_{M-K} G^+ dM < \varepsilon. \tag{2.1.1}$$

For each i, let $\{E_{i,j}\}_j$ be the family of all connected components of

$$M - (\{\exp_p tv ; v \in A_{p_i}, t \geq 0\} \cup D_{p_i}(K)).$$

Each $E_{i,j}$ is an open half plane bounded by two rays emanating from p_i, and has the property that there are no rays emanating from p_i in it. Lemma 1.1 implies that $c(E_{i,j})$ is equal to the inner angle of $E_{i,j}$ at p_i. It follows from (2.1.1) that

$$2\pi - \mathrm{meas}(A_{p_i}) - \theta_i = c\Big(\bigcup_j E_{i,j}\Big) < \varepsilon$$

for all i. Since θ_i tends to zero as $i \to \infty$ and ε is arbitrary, this completes the proof.

Lemma 2.2. *Let $\{p_i\}$ be a divergent sequence of points of M. Assume that for any compact set K, there exists a point $p(K) \in \{p_i\}$ such that $A_{p(K)}(K)$ is empty. Then, there exists a subsequence $\{p_j\}$ of $\{p_i\}$ such that*

$$\lim_{j\to\infty} \operatorname{meas}(A_{p_j}) = 2\pi .$$

Proof. Let $\{K_j\}$ be a monotone divergent sequence of compact sets with $\bigcup K_j = M$ such that $M - K_j$ is an open half cylinder. Set $p_j := p(K_j)$. Then

$$\bigcup D_{p_j}(K_j) = M \quad \text{and} \quad c(D_{p_j}(K_j)) = 2\pi\chi(M) - 2\pi + \theta_j ,$$

where θ_j is the inner angle of $D_{p_j}(K_j)$ at p_j. Thus,

$$c(M) = \lim_{j\to\infty} c(D_{p_j}(K_j)) = 2\pi\chi(M) - 2\pi + \lim_{j\to\infty} \theta_j .$$

Since $2\pi\chi(M) - c(M) \geq 2\pi$, $\lim_{j\to\infty}\theta_j = 0$. Therefore Lemma 2.2 follows from Lemma 2.1.

Lemma 2.3. *For any compact domain $K' \subset M$ such that $M - K'$ is homeomorphic to an open half cylinder, there exists a compact convex domain K bounded by a simple closed curve such that $K' \subset K$.*

Proof. Let Γ be the set of all simple closed curves freely homotopic to $\partial K'$ in $M - \operatorname{Int}(K')$. Let $\{c_i\}$ be a sequence of elements of Γ such that

$$\lim_{i\to\infty} L(c_i) = \inf_{c\in\Gamma} L(c) .$$

For each c_i and for an arbitrary fixed point x_i on c_i there is a $\gamma_i \in \Gamma$ such that

$$L(\gamma_i) = \inf\{L(c)\,; c \in \Gamma, c \text{ passes through } x_i\} .$$

If there exists a subsequence $\{x_j\}$ of $\{x_i\}$ such that $\{x_j\}$ converges to some point in M, then $\lim \gamma_j$ becomes a simple closed curve. The $\lim\ \gamma_j$ bounds a compact convex domain, say, K. If $\{x_i\}$ does not contain any convergent subsequence, then neither does $\{\gamma_i\}$ and hence γ_i is a geodesic loop for all sufficiently large i. Let D_i be a compact domain bounded by γ_i and let θ_i be the inner angle of D_i at x_i. The Gauss-Bonnet Theorem implies

$$c(D_i) = 2\pi\chi(M) - \pi + \theta_i .$$

Since $\bigcup D_i = M$,

$$c(M) = 2\pi\chi(M) - \pi + \lim_{i\to\infty} \theta_i ,$$

which contradicts $2\pi\chi(M) - c(M) \geq 2\pi$. This completes the proof of Lemma 2.3.

Lemma 2.4. *Let $\{p_i\}$ be a divergent sequence of points of M and let K be a compact domain of M such that $M - K$ is homeomorphic to an open half cylinder. Then there exists a number $i(K)$ such that $A'_{p_i}(K)$ is nonempty for all $i \geq i(K)$.*

Proof. Suppose that there exists a divergent sequence $\{p_i\}$ such that all rays emanating from p_i intersect K. From Lemma 2.3, without loss of generality we may assume that K is a convex set and each ray emanating from p_i intersects $\operatorname{Int}(K)$. Take a point p_i with $p_i \in M - K$ and fix it. Assume that ∂K is parametrized positively by arc length relative to K. Let $f_a\colon A_{p_i} \to \partial K$ $(a = 0, 1)$ be the mappings defined by

$$f_a(v) := \exp_{p_i} t_a(\gamma_v) \qquad \text{for } v \in A_{p_i}.$$

Convexity of K implies that there exists a unique minimal subarc J_a^i of ∂K such that $f_a(A_{p_i}) \subset J_a^i$. All rays emanating from p_i pass through points on J_1^i. The endpoints of J_1^i are denoted by $f_1(v_i)$ and $f_1(w_i)$ for $v_i, w_i \in A_{p_i}$. Set $\sigma_i := \gamma_{v_i}$ and $\tau_i := \gamma_{w_i}$. Let D_i' be a domain in $M - K$ homeomorphic to an open half plane whose boundary consists of $\sigma_i|[t_1(\sigma_i), \infty)$, $\tau_i|[t_1(\tau_i), \infty)$ and $\partial K - J_1^i$. By Lemma 1.3 (1) we get a monotone divergent sequence $\{t_j\}$ such that each $S_{t_j}(p_i)$ is a piecewise smooth simple closed curve and $K \subset B_{t_1}(p_i)$. By the choice of D_i', any ray emanating from p_i does not intersect the arc $S_{t_j}(p_i) \cap D_i'$. The same argument as developed in the outline of the proof of Lemma 1.1 implies that there are a cut point q_j to p_i in $S_{t_j}(p_i) \cap D_i'$ and minimizing geodesic segments $\xi_j, \eta_j\colon [0, t_j] \to M$ joining p_i to q_j with $\lim \xi_j = \sigma_i$ and $\lim \eta_j = \tau_i$. Since σ_i and τ_i intersect $\operatorname{Int}(K)$, both ξ_j and η_j for each sufficiently large j intersect $\operatorname{Int}(K)$. Let F_j^i be a disk domain in $M - K$ bounded by $\xi_j|[t_1(\xi_j), t_j]$, $\eta_j|[t_1(\eta_j), t_j]$ and the subarc of ∂K from $\xi_j(t_1(\xi_j))$ to $\eta_j(t_1(\eta_j))$ which is contained entirely in $\partial K - J_1^i$. Let Δ_i be a disk domain in $M - K$ bounded by $\sigma_i|[0, t_0(\sigma_i))$, $\tau_i|[0, t_0(\tau_i))$ and the subarc of ∂K joining $\sigma_i(t_0(\sigma_i))$ and $\tau_i(t_0(\tau_i))$. Set $D_i := D_i' - \operatorname{Cl}(\Delta_i)$ (see Figure 1).

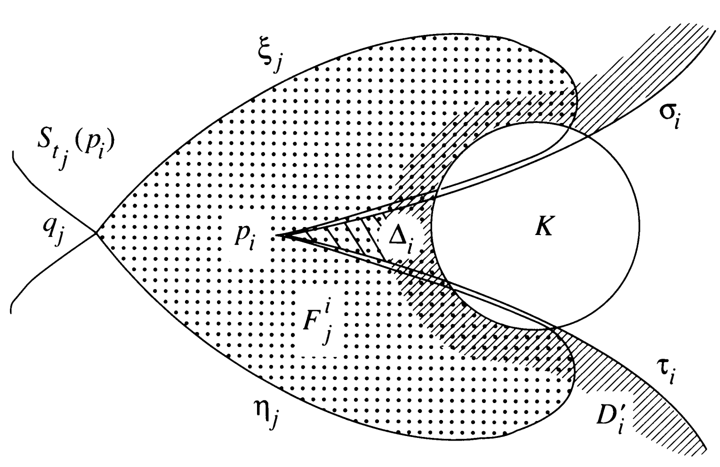

FIGURE 1.

Since $\bigcup_j F_j^i = D_i'$, a slight modification of the discussion in the outline of the proof of Lemma 1.1 implies

$$(2.4.1) \qquad c(D_i) = \pi - \kappa(D_i) \quad \text{and} \quad c(D_i') = \pi - \kappa(D_i').$$

Since σ_i and τ_i intersect K for all i, there are subsequences $\{\sigma_k\}$ and $\{\tau_k\}$ such that $\lim \sigma_k = \sigma$ and $\lim \tau_k = \tau$ for some straight lines σ and τ. Then either $\sigma = \tau$ or $\sigma \cap \tau = \emptyset$. Since K contains finitely many handles, there are two cases for the configuration of four points $\sigma(t_0(\sigma))$, $\sigma(t_1(\sigma))$, $\tau(t_0(\tau))$, $\tau(t_1(\tau))$ on ∂K.

Case 1. *The four points $\sigma(t_0(\sigma))$, $\tau(t_0(\tau))$, $\tau(t_1(\tau))$, $\sigma(t_1(\sigma))$ lie on ∂K in this order.* Choose disjoint two open half planes H_σ and H_τ in $M - K$ such that H_σ (resp. H_τ) is bounded by $\sigma|(-\infty, t_0(\sigma)]$, $\sigma|[t_1(\sigma), \infty)$ and a subarc from $\sigma(t_0(\sigma))$ of ∂K (resp. $\tau((-\infty, t_0(\tau)])$, $\tau([t_1(\tau), \infty))$ and a subarc from $\tau(t_0(\tau))$ to $\tau(t_1(\tau))$ of ∂K). Since the inner angle of D_k at p_k tends to 2π as $k \to \infty$ by Lemma 1.2, we have

$$(2.4.2) \qquad \lim \kappa(D_k) = \kappa(H_\sigma \cup H_\tau) - \pi.$$

On the other hand, $c(H_\sigma) \le -\kappa(H_\sigma)$ and $c(H_\tau) \le -\kappa(H_\tau)$ by the remark (1.4). Hence

$$c(H_\sigma \cup H_\tau) \le -\kappa(H_\sigma \cup H_\tau).$$

For a fixed number $\varepsilon \in (0, \pi/2)$ choose a compact set L as to satisfy

$$(2.4.3) \qquad \int_{M-L} G^+ dM < \varepsilon \quad \text{and} \quad c((H_\sigma \cup H_\tau) \cap L) < -\kappa(H_\sigma \cup H_\tau) + \varepsilon.$$

Then, it follows from (2.4.1) and (2.4.3) that

$$\pi - \kappa(D_k) = c(D_k) < c(D_k \cap L) + \varepsilon < c((H_\sigma \cup H_\tau) \cap L) + 2\varepsilon < -\kappa(H_\sigma \cup H_\tau) + 3\varepsilon$$

for sufficiently large k. Since $\varepsilon < \pi/2$, this contradicts (2.4.2).

Note that the above arguments imply that $\sigma = \tau$ does not occur.

Case 2. *The four points $\sigma(t_0(\sigma))$, $\tau(t_1(\tau))$, $\sigma(t_1(\sigma))$, $\tau(t_0(\tau))$ lie on ∂K in this order.* Let H_σ be the open half plane in $M - K$ bounded by $\sigma|(-\infty, t_0(\sigma)]$, $\sigma|[t_1(\sigma), \infty)$ and by a subarc of ∂K such that $\tau((-\infty, t_0(\tau))) \subset H_\sigma$ and let H_τ be defined similarly. Then, it follows from (1.3) and (1.4) that $c(H_\tau) \le -\kappa(H_\tau)$ and $c(H_\sigma - H_\tau) \le \pi - \kappa(H_\sigma - H_\tau)$. Hence,

$$(2.4.4) \qquad c(H_\sigma \cup H_\tau) \le -\kappa(H_\sigma \cup H_\tau).$$

Let $\varepsilon \in (0, \pi/4)$ be a fixed number, and let L be a compact domain such that

$$\int_{M-L} G^+ dM < \varepsilon.$$

Then, it follows from (2.4.1) that

$$\pi - \kappa(D_k') = c(D_k') < c(D_k' \cap L) + \varepsilon < c((H_\sigma \cup H_\tau) \cap L) + 2\varepsilon$$

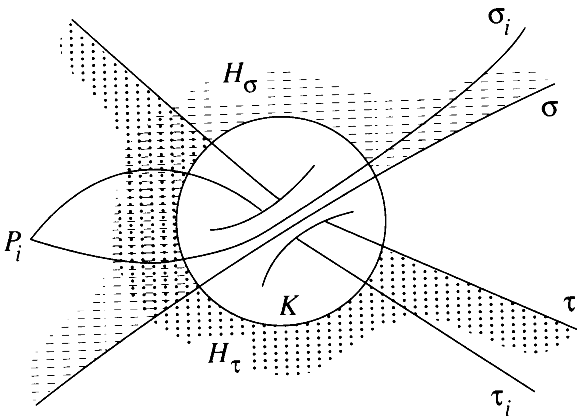

FIGURE 2.

for sufficiently large k. This means that $c(H_\sigma \cup H_\tau)$ is finite. Therefore,

$$\pi - \kappa(D_k') < c(H_\sigma \cup H_\tau) + 3\varepsilon \leq -\kappa(H_\sigma \cup H_\tau) + 3\varepsilon .$$

for all sufficiently large k. Moreover, since σ_k, τ_k tend to σ, τ,

$$\lim \kappa(D_k') = \kappa(H_\sigma \cup H_\tau).$$

This contradicts $\varepsilon < \pi/4$. This completes the proof of Lemma 2.4.

Proof of Theorem A. Let $\{p_i\}$ be an arbitrary divergent sequence of points of M and let K be a compact convex domain as obtained in Lemma 2.3. In view of Lemmas 2.2 and 2.4 we may assume that $A_{p_i}(K)$ and $A'_{p_i}(K)$ are nonempty for all i. Let γ_i, ρ_i be two rays bounding $D_{p_i}(K)$. If $\{p_i\}$ contains a subsequence $\{p_j\}$ such that both $\{\gamma_j\}$ and $\{\rho_j\}$ converge to straight lines, then it follows from Lemmas 1.2 and 2.1 that

$$\lim \operatorname{meas}(A_{p_j}) = 2\pi .$$

Now assume that $\{\gamma_i\}$ does not contain any convergent subsequence. For any i, let σ_i, τ_i be the rays as defined in the proof of Lemma 2.4 and let D_i be the open half plane bounded by $\gamma_i, \sigma_i|[0, t_0(\sigma_i))$, $\sigma_i|[t_1(\sigma_i), \infty)$ and by a subarc of ∂K such that $D_i \subset D_{p_i}(K) - K$. Without loss of generality we may assume that D_i does not contain $\tau_i([t_1(\tau_i), \infty))$. Then there are no rays emanating from p_i in D_i. By a discussion similar to the proof of Lemma 2.4, it follows that

$$c(D_i) = \pi - \kappa(D_i). \tag{2.A.1}$$

Since each σ_i intersects K, there is a subsequence $\{\sigma_j\}$ of $\{\sigma_i\}$ converging to some straight line σ intersecting K. Let H be the open half plane in $M - K$ bounded by $\sigma|(-\infty, t_0(\sigma)]$, $\sigma|[t_1(\sigma), \infty)$ and a subarc from $\sigma(t_1(\sigma))$ to $\sigma(t_0(\sigma))$ of ∂K. Since $\{\gamma_j\}$ does not contain any convergent subsequence, D_j tends to H. By (1.4),

$$c(H) \leq -\kappa(H). \tag{2.A.2}$$

For any positive number ε, there exists a compact set L such that

$$\int_{M-L} G^{+} dM < \varepsilon . \tag{2.A.3}$$

Then it follows from (2.A.1) and (2.A.3) that

$$\pi - \kappa(D_j) = c(D_j) < c(D_j \cap L) + \varepsilon < c(H \cap L) + 2\varepsilon$$

for all sufficiently large j. This means that $c(H)$ is finite. Hence, by (2.A.2),

$$\pi - \kappa(D_j) < c(H \cap L) + 2\varepsilon < c(H) + 3\varepsilon \le -\kappa(H) + 3\varepsilon$$

for all sufficiently large j. On the other hand, if we denote by ψ_j the inner angle of D_j at p_j, then

$$\lim[\kappa(D_j) - \pi + \psi_j] = \kappa(H) .$$

Thus $\psi_j < 4\varepsilon$ for all sufficiently large j. Thus the argument above applies to ρ_i and implies that the inner angle between ρ_i and τ_i at p_i also tends to zero. From Lemma 1.2 the angle between σ_i and τ_i at p_i tends to zero as $i \to \infty$. This completes the proof of Theorem A.

3. The case when M has more than one end

The aim of this section is to prove Theorem B. We assume that M has k ends and admits total curvature.

Proof of Theorem B. Let K be a compact domain and U_i a tube as in §0. Let M_i be a complete open surface with one end such that there exists an isometric embedding $\iota_i : U_i \cup K \to M_i$ and $M_i - \iota_i(U_i \cup K)$ consists of $k-1$ open disk domains. Then the Gauss-Bonnet Theorem implies

$$s_i(M) = 2\pi\chi(M_i) - c(M_i) . \tag{3.B.1}$$

Now, without loss of generality we may assume that K contains a compact domain K' such that $M - K'$ is a disjoint union of k tubes and that $d(M-K, K')$ is greater than the length of $\partial K'$. Then each minimizing segment joining two points in U_i is contained in $K \cup U_i$. For any p in U_i, set

$$A_p(i) := \{v \in A_p \,; \gamma_v([0,\infty)) \subset U_i \cup K\}$$

and let $A_{i,p}$ be the set of all unit vectors at $\iota_i(p)$ tangent to rays emanating from $\iota_i(p)$ in M_i. It follows that the restriction of differential mapping $d\iota_i | A_p(i) : A_p(i) \to A_{i,p}$ is bijective. In particular we have

$$\text{meas}(A_p(i)) = \text{meas}(A_{i,p})$$

for all p in U_i. It follows from the construction of M_i that $A_p(i) \subset A_p = A_p(i) \cup A_p(K)$ and

$$\text{meas}(A_{i,p}) \le \text{meas}(A_p) \le \text{meas}(A_{i,p}) + \text{meas}(A_p(K)) \tag{3.B.2}$$

for all p in U_i. On the other hand, for a divergent sequence $\{p_j\}$ of points in U_i, Theorem A and Lemma 1.2 imply

(3.B.3) $$\lim_{j\to\infty} \operatorname{meas}(A_{i,p_j}) = \min\{2\pi\chi(M_i) - c(M_i), 2\pi\}$$

and

(3.B.4) $$\lim_{j\to\infty} \operatorname{meas}(A_{p_j}(K)) = 0.$$

Therefore, by (3.B.1), (3.B.2), (3.B.3), and (3.B.4),

$$\lim_{j\to\infty} \operatorname{meas}(A_{p_j}) = \lim_{j\to\infty} \operatorname{meas}(A_{i,p_j}) = \min\{s_i(M), 2\pi\}.$$

This completes the proof of Theorem B.

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