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HARMONIC FUNCTIONS HAVING NO TANGENTIAL LIMITS

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ABSTRACT. Let C_0 be a tangential curve in $D = \{|z| < 1\}$ which ends at 1 and let C_θ be its rotation about the origin through an angle θ. We construct a bounded harmonic function in D which fails to have limits along C_θ for all θ, $0 \leq \theta \leq 2\pi$.

1. INTRODUCTION

The classical Fatou theorem asserts that if f is a bounded holomorphic function in the unit disc $D = \{|z| < 1\}$, then f has nontangential boundary values at almost every point on the unit circle $\partial D = \{|z| = 1\}$. The Fatou theorem has been generalized in several directions; a bounded (or more generally, positive) harmonic function in D has nontangential boundary values at almost every point on ∂D. In 1926, Littlewood [3] showed that the Fatou theorem is best possible in the following sense: Let C_0 be an arbitrary tangential curve in D which ends at $z = 1$ and let C_θ be the curve C_0 rotated about the origin through an angle θ, so that C_θ touches ∂D internally at $e^{i\theta}$.

Theorem A. *There exists a bounded harmonic function h in D such that*

$$\lim_{|z|\to 1, z\in C_\theta} h(z)$$

does not exist for almost all θ, $0 \leq \theta \leq 2\pi$. In particular, there exists a bounded holomorphic function f in D such that

$$\lim_{|z|\to 1, z\in C_\theta} f(z)$$

does not exist for almost all θ, $0 \leq \theta \leq 2\pi$.

The second half of Theorem A, in fact, readily follows from the first; if h^* is a harmonic conjugate function of h, then $\exp(h + ih^*)$ is a bounded outer function in D with the property holding for almost all θ. In 1957, Lohwater and Piranian [4] (see also Collingwood and Lohwater [2; Theorem 2.22]) improved the second half of Theorem A by giving the next theorem.

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Theorem B. *There exists a Blaschke product B in D such that*

$$\lim_{|z|\to 1, z\in C_\theta} B(z)$$

does not exist for all θ, $0 \le \theta \le 2\pi$.

Since the logarithm of the modulus of a Blaschke product is a Green potential of a measure distributed on the zeros of the Blaschke product, we may, from the potential theoretical point of view, regard Theorem B as the existence of a Green potential which fails to have limits along C_θ for *all* θ, $0 \le \theta \le 2\pi$. The existence of a bounded harmonic function with the property holding for *all* θ, however, seems to be unknown. In 1980, Barth (see [1; p. 551]) raised a somewhat weaker question. Does there exist a *positive* harmonic function which fails to have limits along C_θ for *all* θ, $0 \le \theta \le 2\pi$?

The purpose of this paper is to show

Theorem. *There exists a bounded harmonic function h in D such that*

$$\lim_{|z|\to 1, z\in C_\theta} h(z)$$

does not exist for all θ, $0 \le \theta \le 2\pi$. In particular, if h^* is a harmonic conjugate function of h, then $\exp(h + ih^*)$ is a bounded outer function in D which fails to have limits along C_θ for all θ, $0 \le \theta \le 2\pi$.

Obviously, the above theorem automatically answers Barth's question in the affirmative. However, we note that Barth's question itself can be solved more easily. We shall, in fact, find a positive *unbounded* harmonic function with the required property (see Remark in §3).

2. Lemmas

For an integrable function f on $[0, 2\pi]$ we let

$$\mathrm{PI}(f, z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\varphi}|^2} f(\varphi)\, d\varphi$$

be the Poisson integral of f. We identify the left end 0 and the right end 2π of the interval $[0, 2\pi]$. If an interval $[\theta_1, \theta_2]$ is not contained in $[0, 2\pi]$, we replace it by

$$(*) \qquad \left\{\Theta = \theta - 2n\pi\,; \theta_1 \le \theta \le \theta_2 \text{ and } n = \left[\frac{\theta}{2\pi}\right]\right\}.$$

Here $[\cdot]$ denotes the greatest integer function. We shall continue to refer to sets defined by $(*)$ as intervals.

Let us begin with some elementary estimates of the Poisson integrals.

Lemma 1. *Let $m > \pi/2$, $0 < c < 1$ and $\eta \in [0, 2\pi]$. Suppose f is measurable on $[0, 2\pi]$, $|f| \le 1$ on $[0, 2\pi]$ and $f(\varphi) = 0$ for $|\varphi - \eta| < mc$. Then*

$$|\mathrm{PI}(f, re^{i\eta})| \le \frac{2}{\pi}\left(\frac{2}{\pi} - \frac{1}{m}\right)^{-2} m^{-1} \qquad \text{for } 1 - c \le r < 1.$$

Proof. Let $1-c \le r < 1$. We observe that if $mc \le |\varphi-\eta| \le \pi$, then

$$|re^{i\eta}-e^{i\varphi}| \ge |e^{i\varphi}-e^{i\eta}|-(1-r) \ge 2\left|\sin\frac{\varphi-\eta}{2}\right| - \frac{1}{m}|\varphi-\eta| \ge \left(\frac{2}{\pi}-\frac{1}{m}\right)|\varphi-\eta|.$$

Hence

$$|\operatorname{PI}(f, re^{i\eta})| \le \frac{1-r^2}{2\pi}\left(\frac{2}{\pi}-\frac{1}{m}\right)^{-2}\int_{|\varphi-\eta|\ge mc}\frac{d\varphi}{|\varphi-\eta|^2},$$

which implies the desired inequality.

Lemma 2 (cf. [3; Lemma 2]). *Let $m_1 > \pi/2$ be a constant such that*

$$\frac{2}{\pi}\left(\frac{2}{\pi}-\frac{1}{m_1}\right)^{-2} m_1^{-1} \le \frac{1}{4}.$$

Let $0 < c < 1$ and $\eta \in [0, 2\pi]$. Suppose f is measurable on $[0, 2\pi]$, $|f| \le 1$ on $[0, 2\pi]$ and $f(\varphi) = 1$ for $|\varphi-\eta| < m_1 c$. Then

$$\operatorname{PI}(f, re^{i\eta}) \ge \frac{1}{2} \qquad \textit{for } 1-c \le r < 1.$$

Proof. We write $\operatorname{PI}(f, re^{i\eta}) = 1 + 2\operatorname{PI}(\frac{1}{2}(f-1), re^{i\eta})$ and apply Lemma 1 to $\frac{1}{2}(f-1)$ to obtain

$$\operatorname{PI}(f, re^{i\eta}) \ge 1 - 2\cdot\frac{2}{\pi}\left(\frac{2}{\pi}-\frac{1}{m_1}\right)^{-2} m_1^{-1} \ge \frac{1}{2}.$$

Lemma 3. *Let $0 < \varepsilon < \frac{1}{4}$ and $0 < c < 1$. Suppose f is measurable on $[0, 2\pi]$, $|f| \le 1$ on $[0, 2\pi]$ and*

$$c^{-1}\int_{|\varphi-\eta|<c} |f(\varphi)|\, d\varphi \le \varepsilon \tag{1}$$

for all $\eta \in [0, 2\pi]$. Then

$$\sup_{|z|\le 1-c} |\operatorname{PI}(f, z)| \le m_2\sqrt{\varepsilon},$$

where $m_2 = 2/\pi + (2/\pi)(2/\pi - 1/2)^{-2}$.

Proof. Let $r = 1-c$ and take an arbitrary point $re^{i\eta}$ on the circle $\{|z| = r\}$. We decompose f into $f_1 + f_2$, where $f_1 = f\chi_{|\varphi-\eta|<c/\sqrt{\varepsilon}}$. Then

$$\begin{aligned}|\operatorname{PI}(f_1, re^{i\eta})| &\le \frac{1}{2\pi}\int_{|\varphi-\eta|<c/\sqrt{\varepsilon}} \frac{1-r^2}{(r-1)^2}|f(\varphi)|\, d\varphi \\ &\le \frac{1}{2\pi}\cdot\frac{2}{c}\cdot c\varepsilon(1+[1/\sqrt{\varepsilon}]) \le \frac{2}{\pi}\sqrt{\varepsilon},\end{aligned}$$

where $[1/\sqrt{\varepsilon}]$ stands for the greatest integer not greater than $1/\sqrt{\varepsilon}$. Here the second inequality follows from (1) and the fact that the interval $\{|\varphi-\eta| < c/\sqrt{\varepsilon}\}$

of length $2c/\sqrt{\varepsilon}$ is covered by at most $1+[1/\sqrt{\varepsilon}]$ intervals of length $2c$. Applying Lemma 1 to $f=f_2$ and $m=1/\sqrt{\varepsilon}$, we obtain

$$|\mathrm{PI}(f_2, re^{i\eta})| \leq \frac{2}{\pi}\left(\frac{2}{\pi}-\sqrt{\varepsilon}\right)^{-2}\sqrt{\varepsilon} \leq \frac{2}{\pi}\left(\frac{2}{\pi}-\frac{1}{2}\right)^{-2}\sqrt{\varepsilon}.$$

Therefore $|\mathrm{PI}(f, z)| \leq m_2\sqrt{\varepsilon}$ on the circle $\{|z|=1-c\}$, and hence the maximum principle proves the lemma.

We define the set valued mapping T from D to $[0, 2\pi]$ by

$$Tz = \{\theta; z \in C_\theta\}.$$

If none of C_θ contains z, then $Tz=\varnothing$. Let M be a subset of D. We observe that $T(M)=[0, 2\pi]$ if and only if every curve C_θ, $0\leq\theta\leq 2\pi$, meets the set M. Let γ be a subcurve of C_0 and let γ_θ be its rotation about the origin through θ. Therefore $\gamma=\gamma_0$. We define the set valued mapping T_γ by

$$T_\gamma z = \{\theta; z\in\gamma_\theta\}.$$

Obviously, $T_\gamma z \subset Tz$. We denote by γ^* the radial projection of γ onto ∂D. It follows from the connectedness of γ that γ^* is a circular interval on ∂D or a singleton set. If γ contains both the end points, then γ^* is a closed circular interval or a singleton set. We denote by $l(\gamma^*)$ the length of γ^*. Note that the curve γ itself may not be rectifiable.

Lemma 4. *Let γ be a subcurve of C_0 connecting $ae^{i\alpha}$ and $be^{i\beta}$, $0<a<b\leq 1$, such that if $z\in\gamma$, then $a\leq|z|\leq b$. Put $M(\eta)=\{re^{i\eta}; a\leq r\leq b\}$. Then $T_\gamma(M(\eta))$ is a closed interval of length $l(\gamma^*)$. More precisely, if $\gamma^*=\{e^{i\theta}; \theta_1\leq \theta\leq\theta_2\}$, then $T_\gamma(M(\eta))=[\eta-\theta_2, \eta-\theta_1]$.*

Proof. Suppose that $\gamma^*=\{e^{i\theta}; \theta_1\leq\theta\leq\theta_2\}$. For each $\theta\in[\theta_1, \theta_2]$ we find r_θ, $a\leq r_\theta\leq b$, such that $r_\theta e^{i\theta}\in\gamma$, or equivalently $r_\theta e^{i\eta}\in\gamma_{\eta-\theta}$. Hence $\eta-\theta\in T_\gamma(M(\eta))$, so that $[\eta-\theta_2, \eta-\theta_1]\subset T_\gamma(M(\eta))$. Conversely, if $\theta\in T_\gamma(M(\eta))$, then there is r_θ, $a\leq r_\theta\leq b$, such that $r_\theta e^{i\eta}\in\gamma_\theta$. Hence $r_\theta e^{i(\eta-\theta)}\in\gamma$, so that $\theta_1\leq\eta-\theta\leq\theta_2$, or equivalently $\theta\in[\eta-\theta_2, \eta-\theta_1]$. Therefore $T_\gamma(M(\eta))\subset[\eta-\theta_2, \eta-\theta_1]$.

Since C_0 is tangential, it follows that C_0 eventually lies outside any Stoltz region with vertex at the point 1, so that

$$\frac{l(C_0(z)^*)}{1-|z|}\to\infty$$

as $z\in C_0$ tends to 1, where $C_0(z)$ is the subcurve of C_0 that connects the points z and 1. We have from this fact

Lemma 5. *For each $m>1$ we can choose a sequence of subcurves γ_j of C_0 with the following properties:*

(i) *γ_j connects $a_je^{i\alpha_j}$ and $b_je^{i\beta_j}$.*

(ii) *If* $z \in \gamma_j$, *then* $a_j \leq |z| \leq b_j$.
(iii) $l(\gamma_j^*) > j(1-a_j)$.
(iv) $0 < 1-b_j < 1-a_j < (1-b_{j-1})/m < (1-a_{j-1})/m$.

Proof. We shall choose γ_j inductively. Assume we have already gotten $\gamma_0, \dots, \gamma_{j-1}$ with the properties (i)–(iv). By the observation before the lemma we can take a_j such that

$$0 < 1-a_j < (1-b_{j-1})/m, \qquad l(\gamma'^*_l) > 2j(1-a_j),$$

where γ'_j is the subcurve of C_0 that connects the point 1 and the point $a_j e^{i\alpha_j}$ at which C_0 meets the circle $\{|z| = a_j\}$ for the last time. Let γ''_j be a subcurve of γ'_j which connects $a_j e^{i\alpha_j}$ and a point near 1 such that $l(\gamma''^*_j) > \frac{1}{2} l(\gamma'^*_j)$. Now take b_j such that

$$\sup_{z \in \gamma''_j} |z| < b_j < 1.$$

Let γ_j be the subcurve of C_0 that connects $a_j e^{i\alpha_j}$ and the point $b_j e^{i\beta_j}$ at which C_0 meets the circle $\{|z| = b_j\}$ for the first time. Then we see that $0 < 1-b_j < 1-a_j$ and that $a_j \leq |z| \leq b_j$ for $z \in \gamma_j$. Moreover γ''_j is included in γ_j, so that

$$l(\gamma_j^*) > j(1-a_j).$$

Thus we can choose a sequence of subcurves γ_j with the properties (i)–(iv).

3. Proof of theorem

Proof of theorem. For $m = m_1$, the constant appearing in Lemma 2, we choose a sequence of subcurves γ_j as in Lemma 5. For a moment we fix j. Let a_j and b_j be as in Lemma 5. By $M_j(\eta)$ we denote the radial line segment $\{re^{i\eta}; a_j \leq r \leq b_j\}$. We obtain from Lemma 4 applied to $\gamma = \gamma_j$ and $M(\eta) = M_j(\eta)$ that $T_{\gamma_j}(M_j(\eta))$ is a closed interval of length $l(\gamma_j^*)$, where we recall that T_{γ_j} is the set valued mapping defined before Lemma 4. Let N be the integer such that

$$2\pi / l(\gamma_j^*) \leq N < 1 + 2\pi / l(\gamma_j^*) \tag{2}$$

and let $\eta_k = 2\pi k/N$. Since $|\eta_k - \eta_{k+1}| \leq l(\gamma_j^*)$, it follows that

$$[0, 2\pi] = \bigcup_{k=1}^{N} T_{\gamma_j}(M_j(\eta_k)),$$

so that $M_j = \bigcup_{k=1}^{N} M_j(\eta_k)$ satisfies $T(M_j) = [0, 2\pi]$ by the fact that $T_{\gamma_j} z \subset Tz$ for $z \in D$. Let

$$I_k = [\eta_k - m_1(1-a_j), \eta_k + m_1(1-a_j)],$$

$$E_j = \bigcup_{k=1}^{N} I_k,$$

$$g_j = \chi_{E_j}.$$

Secondly, we vary the index j and extract a subsequence of γ_j satisfying the following (3) and (4). We see from (2) that $|E_j| = 2m_1(1-a_j)N \le 2m_1(1-a_j)\cdot 4\pi/l(\gamma_j^*)$, so that $|E_j|$ tends to 0 by (iii) of Lemma 5. Hence, taking a subsequence of γ_j if necessary, we may assume that

$$\sum_{j=1}^{\infty} |E_j| < \infty. \tag{3}$$

We claim that g_j satisfies an inequality of type (1). We observe from (2) that an interval of length $4(1-b_{j-1})$ in $[0, 2\pi]$ contains at most ν points η_k, where

$$\nu \le 1 + \frac{4(1-b_{j-1})}{2\pi/N} \le \frac{4}{\pi}(1-b_{j-1})\left(1+\frac{2\pi}{l(\gamma_j^*)}\right) \le 16\frac{1-b_{j-1}}{l(\gamma_j^*)}.$$

The length of I_k is equal to $2m_1(1-a_j) < 2(1-b_{j-1})$ by (iv) of Lemma 5. Hence if I is an interval of length $2(1-b_{j-1})$ in $[0, 2\pi]$, then the number of I_k which intersect I is bounded by $16(1-b_{j-1})/l(\gamma_j^*)$. Therefore g_j satisfies

$$\frac{1}{1-b_{j-1}}\int_{|\varphi-\eta|<1-b_{j-1}} g_j(\varphi)\,d\varphi \le 32m_1\frac{1-a_j}{l(\gamma_j^*)}$$

for all $\eta \in [0, 2\pi]$. Since the right hand side tends to 0 by (iii) of Lemma 5, it follows from Lemma 3 with $f = g_j$ and $c = 1 - b_{j-1}$ that

$$\sup_{|z|\le b_{j-1}} \mathrm{PI}(g_j, z) \le m_2\sqrt{32m_1}\sqrt{(1-a_j)/l(\gamma_j^*)}$$

for sufficiently large j. Hence, taking a subsequence of γ_j if necessary, we may assume that

$$\sup_{|z|\le b_{j-1}} \mathrm{PI}(g_j, z) \le 9^{-j}. \tag{4}$$

Thirdly, we form two sequences $\{F_j\}$ and $\{G_j\}$ of sets inductively. Let $F_1 = \varnothing$, $G_1 = E_1$ and let

$$F_{2j} = F_{2j-1} \cup E_{2j}, \qquad G_{2j} = G_{2j-1}\setminus E_{2j},$$
$$F_{2j+1} = F_{2j}\setminus E_{2j+1}, \qquad G_{2j+1} = G_{2j} \cup E_{2j+1}$$

for $j \ge 1$. We see that $F_j \cap G_j = \varnothing$, $F_j \cup G_j = \bigcup_{k=1}^{j} E_k$ and $E_j \subset F_j$ if j is even; $E_j \subset G_j$ if j is odd. Let

$$f_j = \begin{cases} 1 & \text{on } F_j, \\ -1 & \text{on } G_j, \\ 0 & \text{elsewhere} \end{cases}$$

We observe that if $\varphi \notin \limsup E_j$, then $f_j(\varphi)$ converges to the limit

$$f(\varphi) = \begin{cases} 1 & \text{if } \varphi \in E_j \text{ with even } j \text{ for the last time}, \\ -1 & \text{if } \varphi \in E_j \text{ with odd } j \text{ for the last time}, \\ 0 & \text{if } \varphi \notin \bigcup_{j=1}^{\infty} E_j. \end{cases}$$

It follows from (3) that $\limsup E_j$ is of measure zero, so that f_j converges to f almost everywhere on $[0, 2\pi]$. We note that

(5) $$f_j = (-1)^j \quad \text{on } E_j,$$

(6) $$|f_j| \le 1 \quad \text{on } [0, 2\pi],$$

(7) $$|f_j - f_{j+1}| \le 2g_{j+1} \quad \text{on } [0, 2\pi].$$

We are now in the final stage. We claim that $h(z) = \mathrm{PI}(f, z)$ is the required bounded harmonic function. It is clear that $|h| \le 1$ on D. Let $h_j(z) = \mathrm{PI}(f_j, z)$. Note that h_j converges to h uniformly on every compact set in D by the dominated convergence theorem. We infer from (5), (6) and Lemma 2 with $f = f_j$, $\eta = \eta_k$ and $c = 1 - a_j$ that if j is even, then

$$h_j(z) \ge \tfrac{1}{2} \quad \text{for } z \in M_j;$$

if j is odd, then

$$h_j(z) \le -\tfrac{1}{2} \quad \text{for } z \in M_j.$$

We infer from (4) and (7) that

$$\sup_{|z| \le b_j} |h_j(z) - h_{j+1}(z)| \le 2 \sup_{|z| \le b_j} \mathrm{PI}(g_{j+1}, z) \le 2/9^{j+1}.$$

Hence

$$\sum_{k=j}^{\infty} \sup_{|z| \le b_j} |h_k(z) - h_{k+1}(z)| \le \sum_{k=j}^{\infty} \frac{2}{9^{k+1}} \le \frac{1}{4},$$

Therefore, if j is even, then

$$h(z) = h_j(z) + (h_{j+1}(z) - h_j(z)) + \cdots \ge \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4} \quad \text{for } z \in M_j;$$

if j is odd, then

$$h(z) \le -\tfrac{1}{4} \quad \text{for } z \in M_j.$$

Since $T(M_j) = [0, 2\pi]$, it follows that if j is even, then

$$\sup_{a_j \le |z| \le b_j, z \in C_\theta} h(z) \ge \tfrac{1}{4} \quad \text{for all } \theta \in [0, 2\pi];$$

if j is odd, then

$$\inf_{a_j \le |z| \le b_j, z \in C_\theta} h(z) \le -\tfrac{1}{4} \quad \text{for all } \theta \in [0, 2\pi].$$

Consequently,

$$\liminf_{|z| \to 1, z \in C_\theta} h(z) \le -\tfrac{1}{4} < \tfrac{1}{4} \le \limsup_{|z| \to 1, z \in C_\theta} h(z) \quad \text{for all } \theta \in [0, 2\pi].$$

Thus this h satisfies the required property. The proof is complete.

Remark. Let E_j be as in the above proof. Since $|E_j| \to 0$, we can take a nonnegative sequence $\{p_j\}$ such that $\limsup_{j \to \infty} p_j = \infty$ and $f = \sum_{j=1}^{\infty} p_j \chi_{E_j}$ is integrable. In view of Lemma 2 and $T(M_j) = [0, 2\pi]$, we obtain that $h = \mathrm{PI}(f, z)$ is a positive harmonic function such that

$$\limsup_{|z| \to 1, z \in C_\theta} h(z) = \infty \quad \text{for all } \theta \in [0, 2\pi].$$

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