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SOME PROPERTIES OF K-SEMISTRATIFIABLE SPACES

T. MIZOKAMI

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ABSTRACT. We study spaces admitting semistratification and k-semistratifications with (CF) property. The class of k-semistratifiable spaces with (CF) property lies between the class of Lašnev spaces and that of k-semistratifiable spaces, and really differs from the classes of stratifiable spaces and $\aleph$-spaces.

1. INTRODUCTION

All spaces are assumed to be regular Hausdorff topological spaces. The letter τ denotes the topology of a space X. We denote by the letter ω the set of all positive integers.

In his paper [8], Lutzer introduced the class of k-semistratifiable spaces, which lies between the class of stratifiable spaces in the sense of Borges [1] and Ceder [2] and the class of semistratifiable spaces introduced by Michael and studied by Creede. The class of σ-spaces introduced by Okuyama lies between that of stratifiable spaces and that of semistratifiable spaces. In this paper, we consider the limited classes of k-semistratifiable and semistratifiable spaces with (CF) property defined below. We give a few characterizations of Lašnev spaces in terms of k-semistratifiable spaces and CF families which are introduced here.

Throughout this paper, σ-spaces are spaces with a σ-discrete network or equivalently, σ-closure-preserving network, and $\aleph$-spaces are spaces with a σ-locally finite k-network. Stratifiable spaces are spaces with the stratification. As for the definition of stratifications, refer to Borges [1].

2. K-SEMISTRATIFIABLE SPACES WITH (CF) PROPERTY

We state the original definition of k-semistratifiable spaces.

Definition 1 (Lutzer [8]). A space X is called a *k-semistratifiable space* if there exists a function $S: \omega \times \tau \to$ {closed subsets of X} such that:

(a) For each $U \in \tau, U = \bigcup\{S(n, U): n \in \omega\}$.
(b) If U, $V \in \tau$ and $U \subset V$, then $S(m, U) \in S(m, V)$ for each m.

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(c) If $C \subset U \in \tau$ with C compact, then $C \subset S(m, U)$ for some m. (We call S a *k-semistratification of X*.)

Definition 2 ([10, Definition 3.1]). A family $\mathscr{U}$ of subsets of a space X is called *finite on compact subsets* of X, briefly CF *in* X, if $\mathscr{U}/K$ is a finite family for any compact subsets K of X.

Definition 3. A semistratification or k-semistratification S of a space X is called to *have* (CF) *property* if the following condition (CF) is satisfied:

(CF) For each $n \in \omega$, $\{S(n, U) \colon U \in \tau\}$ is CF in X.

A space having S with (CF) property is called a *semistraifiable* or a *k-semistratifiable space with* (CF) *property*, respectively.

Theorem 1. *If a space X has a σ-HCP* (= *hereditarily closure-preserving*) *k-network, then X is a k-semistratifiable space with* (CF) *property.*

Proof. Let $\bigcup\{\mathscr{H}_n \colon n \in \omega\}$ be a k-network for X, where, for each n, $\mathscr{H}_n \subset \mathscr{H}_{n+1}$ and $\mathscr{H}_n$ is an HCP family of closed subsets of X. For each $(n, U) \in \omega \times \tau$, let

$$S(n, U) = \bigcup\{H \in \mathscr{H}_n \colon H \subset U\}.$$

Then it is easily seen from [10, Proposition 3.2] that S is a k-semistratification with (CF) property.

Example 1. There exists a stratifiable, k-semistratifiable space with (CF) property, but does not have a σ-HCP k-network.

Proof. Let Y be a non-metrizable Lašnev space which has no σ-locally finite k-network. (For example, let Y be the quotient space obtained from $\bigoplus\{S_\alpha \colon \alpha < \omega_1\}$ by identifying all the limit points, where each S_α is the convergent sequence with its limit point.) Then by [6] the product space $X = Y \times [0, 1]$ has no σ-HCP k-network. X is obviously a stratifiable space. By Theorem 5, stated later, X is a k-semistratifiable space with (CF) property.

Theorem 2. *For a space X, the following are equivalent*:

(1) *X is a Lašnev space.*
(2) *X is a Fréchet, k-semistratifiable space with* (CF) *property.*
(3) *X is a Fréchet space which has a σ-*CF *pseudobase.*

Proof. (1) $\to$ (2) follows from [3] and Theorem 1.
(2) $\to$ (3). Let S be the k-semistratification of X with (CF) property. Then

$$\bigcup[\{S(n, U) \colon U \in \tau\} \colon n \in \omega\}]$$

is a σ-CF pseudobase of X.
(3) $\to$ (1) follows from [10, Theorem 4.1, (9)].

Corollary. *A space X is metrizable if and only if X is a first countable, k-semistratifiable space with* (CF) *property.*

We notice that a Lašnev space cannot be characterized to be a Fréchet space with a σ-HCP "pseudobase" [5]. For the next example, we prepare a lemma.

Lemma. *If a space X has a σ-HCP k-network $\mathscr{H}$ of closed subsets of X, then $X = X_1 \cup X_2$, where X_1 is a σ-discrete closed subspace and X_2 is an $\aleph$-space such that for each $p \in X_2$, $\mathscr{H}$ is σ-locally finite at p in X.*

Proof. Let $\mathscr{H} = \bigcup\{\mathscr{H}_n : n \in \omega\}$, where for each n $\mathscr{H}_n \subset \mathscr{H}_{n+1}$ and $\mathscr{H}_n$ is an HCP family of closed subsets of X. Let

$$X_1 = \{p \in X : \bigcap\{H \in \mathscr{H}_n : p \in H\} \text{ is a finite subset for some } n\}.$$

Then by the same argument as in [11, Theorem 3.6], we can show that X_1 is a countable union of discrete closed subsets of X and $X_2 = X - X_1$ has the required property.

Example 2. There exists a stratifiable space which has no σ-HCP k-network.

Proof. For each $\alpha < \omega_1$, let T_α be the copy of the subspace $T = \{(x, y) : 0 \leq x, y \leq 1\}$ of $\mathbf{R}^2$ and $f_\alpha : T \to T_\alpha$ its homeomorphism. Let X be the quotient space obtained from $\bigoplus\{T_\alpha : \alpha < \omega_1\}$ by identifying $\{f_\alpha((x, 0) : \alpha < \omega_1\}$ for each x with $0 \leq x \leq 1$. Since X is dominated by metric spaces, X is a stratifiable space [1, Theorem 7.2]. If X has a σ-HCP k-network $\mathscr{H}$, then by the above, there exists a point $p = f(f_\alpha((x, 0))) \in X$ such that $\mathscr{H}$ is σ-locally finite at p in X, where $f : \bigoplus T_\alpha \to X$ is the quotient mapping. But, by [7, Remark 2] this is a contradiction.

Example 3. There exists a k-semistratifiable space which does not have (CF) property.

Proof. Let X be the same space as in [2, Example 9.2]. Then X is a first countable, non-metrizable stratifiable space. By the Corollary to Theorem 2, X has no k-semistratification with (CF) property.

From the argument as in Theorem 1, the following is easily seen.

Theorem 3. *Any σ-space is a semistratifiable space with* (CF) *property.*

The converse is not known. However, we have a partial answer to it.

Theorem 4. *If a space X is a Fréchet, semistratifiable space with* (CF) *property, then X is a σ-space.*

Proof. It is easy to see that X has a σ-CF network $\mathscr{H}$. By [10, Proposition 3.3], $\mathscr{H}$ is a σ-closure-preserving network. Hence X is a σ-space.

The following example shows that any semistratifiable space need not have (CF) property.

Example 4. There exists a first countable, semistratifiable space which is not a σ-space.

Proof. Let X be the space in [4, Example 9.10]. Since X is not a σ-space, by Theorem 4, X is not a semistratifiable space with (CF) property.

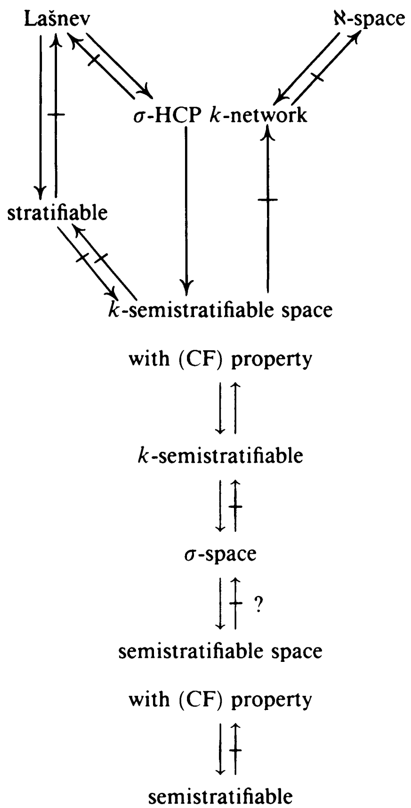
Theorem 5. *If a space X is embedded into a countable product of Lašnev spaces, then X is a k-semistratifiable space with* (CF) *property.*

Proof. By the same method as in [9, Lemma 5.1 and Proposition 6.1] and by [10, Proposition 3.3], we can show that X has a σ-closure-preserving, CF family $\bigcup_n \mathscr{H}_n$ of closed subsets of X, which forms a k-network for X. For each $(n, U) \in \omega \times \tau$, let

$$S(n, U) = \bigcup \left\{ H \in \bigcup_{t \leq n} \mathscr{H}_t : H \subset U \right\}.$$

Then S is a k-semistratification with (CF) property.

The other known implications are indicated by the diagram below. The remaining proofs are easy and well-known, and therefore they are omitted.



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JOETSU UNIVERSITY OF EDUCATION, JOETSU, NIIGATA 943, JAPAN