SOME PROPERTIES OF K-SEMISTRATIFIABLE SPACES

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Abstract. We study spaces admitting semistratification and k-semistratifications with (CF) property. The class of k-semistratifiable spaces with (CF) property lies between the class of Lašnev spaces and that of k-semistratifiable spaces, and really differs from the classes of stratifiable spaces and N-spaces.

1. Introduction

All spaces are assumed to be regular Hausdorff topological spaces. The letter \(\tau\) denotes the topology of a space \(X\). We denote by the letter \(\omega\) the set of all positive integers.

In his paper [8], Lutzer introduced the class of k-semistratifiable spaces, which lies between the class of stratifiable spaces in the sense of Borges [1] and Ceder [2] and the class of semistratifiable spaces introduced by Michael and studied by Creede. The class of \(\sigma\)-spaces introduced by Okuyama lies between that of stratifiable spaces and that of semistratifiable spaces. In this paper, we consider the limited classes of k-semistratifiable and semistratifiable spaces with (CF) property defined below. We give a few characterizations of Lašnev spaces in terms of k-semistratifiable spaces and CF families which are introduced here.

Throughout this paper, \(\sigma\)-spaces are spaces with a \(\sigma\)-discrete network or equivalently, \(\sigma\)-closure-preserving network, and N-spaces are spaces with a \(\sigma\)-locally finite \(k\)-network. Stratifiable spaces are spaces with the stratification. As for the definition of stratifications, refer to Borges [1].

2. K-SEMISTRATIFIABLE SPACES WITH (CF) PROPERTY

We state the original definition of k-semistratifiable spaces.

Definition 1 (Lutzer [8]). A space \(X\) is called a k-semistratifiable space if there exists a function \(S: \omega \times \tau \rightarrow \{\text{closed subsets of } X\}\) such that:

(a) For each \(U \in \tau, U = \bigcup\{S(n, U) : n \in \omega\}\).

(b) If \(U, V \in \tau\) and \(U \subseteq V\), then \(S(m, U) \subseteq S(m, V)\) for each \(m\).

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(c) If \( C \subset U \in \tau \) with \( C \) compact, then \( C \subset S(m, U) \) for some \( m \). (We call \( S \) a \( k \)-semistratification of \( X \).

**Definition 2** ([10, Definition 3.1]). A family \( \mathcal{U} \) of subsets of a space \( X \) is called **finite on compact subsets of \( X \)**, briefly \( \text{CF in } X \), if \( \mathcal{U}/K \) is a finite family for any compact subsets \( K \) of \( X \).

**Definition 3.** A semistratification or \( k \)-semistratification \( S \) of a space \( X \) is called to have (CF) property if the following condition (CF) is satisfied:

(CF) For each \( n \in \omega \), \( \{S(n, U) : U \in \tau\} \) is \( \text{CF in } X \).

A space having \( S \) with (CF) property is called a semistraifiable or a \( k \)-semistratifiable space with (CF) property, respectively.

**Theorem 1.** If a space \( X \) has a \( \sigma \)-HCP (= hereditarily closure-preserving) \( k \)-network, then \( X \) is a \( k \)-semistratifiable space with (CF) property.

**Proof.** Let \( \bigcup \{\mathcal{H}_n : n \in \omega\} \) be a \( k \)-network for \( X \), where, for each \( n \), \( \mathcal{H}_n \subset \mathcal{H}_{n+1} \) and \( \mathcal{H}_n \) is an HCP family of closed subsets of \( X \). For each \( (n, U) \in \omega \times \tau \), let

\[
S(n, U) = \bigcup \{H \in \mathcal{H}_n : H \subset U\}.
\]

Then it is easily seen from [10, Proposition 3.2] that \( S \) is a \( k \)-semistratification with (CF) property.

**Example 1.** There exists a stratifiable, \( k \)-semistratifiable space with (CF) property, but does not have a \( \sigma \)-HCP \( k \)-network.

**Proof.** Let \( Y \) be a non-metrizable Lašnev space which has no \( \sigma \)-locally finite \( k \)-network. (For example, let \( Y \) be the quotient space obtained from \( \bigoplus \{S_\alpha : \alpha < \omega_1\} \) by identifying all the limit points, where each \( S_\alpha \) is the convergent sequence with its limit point.) Then by [6] the product space \( X = Y \times [0, 1] \) has no \( \sigma \)-HCP \( k \)-network. \( X \) is obviously a stratifiable space. By Theorem 5, stated later, \( X \) is a \( k \)-semistratifiable space with (CF) property.

**Theorem 2.** For a space \( X \), the following are equivalent:

1. \( X \) is a Lašnev space.
2. \( X \) is a Fréchet, \( k \)-semistratifiable space with (CF) property.
3. \( X \) is a Fréchet space which has a \( \sigma \)-CF pseudobase.

**Proof.** (1) \( \rightarrow \) (2) follows from [3] and Theorem 1.

(2) \( \rightarrow \) (3). Let \( S \) be the \( k \)-semistratification of \( X \) with (CF) property. Then

\[
\bigcup \{\{S(n, U) : U \in \tau\} : n \in \omega\}
\]

is a \( \sigma \)-CF pseudobase of \( X \).

(3) \( \rightarrow \) (1) follows from [10, Theorem 4.1, (9)].

**Corollary.** A space \( X \) is metrizable if and only if \( X \) is a first countable, \( k \)-semistratifiable space with (CF) property.

We notice that a Lašnev space cannot be characterized to be a Fréchet space with a \( \sigma \)-HCP “pseudobase” [5]. For the next example, we prepare a lemma.
Lemma. If a space $X$ has a $\sigma$-HCP $k$-network $\mathcal{H}$ of closed subsets of $X$, then $X = X_1 \cup X_2$, where $X_1$ is a $\sigma$-discrete closed subspace and $X_2$ is an $\mathcal{H}$-space such that for each $p \in X_2$, $\mathcal{H}$ is $\sigma$-locally finite at $p$ in $X$.

Proof. Let $\mathcal{H} = \bigcup\{\mathcal{H}_n : n \in \omega\}$, where for each $n$, $\mathcal{H}_n \subset \mathcal{H}_{n+1}$ and $\mathcal{H}_n$ is an HCP family of closed subsets of $X$. Let

$$X_1 = \{p \in X : \bigcap\{H \in \mathcal{H}_n : p \in H\} \text{ is a finite subset for some } n\}.$$

Then by the same argument as in [11, Theorem 3.6], we can show that $X_1$ is a countable union of discrete closed subsets of $X$ and $X_2 = X - X_1$ has the required property.

Example 2. There exists a stratifiable space which has no $\sigma$-HCP $k$-network.

Proof. For each $\alpha < \omega_1$, let $T_\alpha$ be the copy of the subspace $T = \{(x, y) : 0 \leq x, y \leq 1\}$ of $\mathbb{R}^2$ and $f_\alpha : T \to T_\alpha$ its homeomorphism. Let $X$ be the quotient space obtained from $\bigoplus\{T_\alpha : \alpha < \omega_1\}$ by identifying $\{f_\alpha((x, 0)) : \alpha < \omega_1\}$ for each $x$ with $0 < x < 1$. Since $X$ is dominated by metric spaces, $X$ is a stratifiable space [1, Theorem 7.2]. If $X$ has a $\sigma$-HCP $k$-network $\mathcal{H}$, then by the above, there exists a point $p = f((x, 0))) \in X$ such that $\mathcal{H}$ is $\sigma$-locally finite at $p$ in $X$, where $f : \bigoplus T_\alpha \to X$ is the quotient mapping. But, by [7, Remark 2] this is a contradiction.

Example 3. There exists a $k$-semistratifiable space which does not have $(CF)$ property.

Proof. Let $X$ be the same space as in [2, Example 9.2]. Then $X$ is a first countable, non-metrizable stratifiable space. By the Corollary to Theorem 2, $X$ has no $k$-semistratification with $(CF)$ property.

From the argument as in Theorem 1, the following is easily seen.

Theorem 3. Any $\sigma$-space is a semistratifiable space with $(CF)$ property.

The converse is not known. However, we have a partial answer to it.

Theorem 4. If a space $X$ is a Fréchet, semistratifiable space with $(CF)$ property, then $X$ is a $\sigma$-space.

Proof. It is easy to see that $X$ has a $\sigma$-CF network $\mathcal{H}$. By [10, Proposition 3.3], $\mathcal{H}$ is a $\sigma$-closure-preserving network. Hence $X$ is a $\sigma$-space.

The following example shows that any semistratifiable space need not have $(CF)$ property.

Example 4. There exists a first countable, semistratifiable space which is not a $\sigma$-space.

Proof. Let $X$ be the space in [4, Example 9.10]. Since $X$ is not a $\sigma$-space, by Theorem 4, $X$ is not a semistratifiable space with $(CF)$ property.
**Theorem 5.** If a space $X$ is embedded into a countable product of Lašnev spaces, then $X$ is a $k$-semistratifiable space with (CF) property.

**Proof.** By the same method as in [9, Lemma 5.1 and Proposition 6.1] and by [10, Proposition 3.3], we can show that $X$ has a $\sigma$-closure-preserving, CF family $\bigcup_n \mathcal{H}_n$ of closed subsets of $X$, which forms a $k$-network for $X$. For each $(n, U) \in \omega \times \tau$, let

$$S(n, U) = \bigcup \left\{ H \in \bigcup_{i \leq n} \mathcal{H}_i : H \subset U \right\}.$$

Then $S$ is a $k$-semistratification with (CF) property.

The other known implications are indicated by the diagram below. The remaining proofs are easy and well-known, and therefore they are omitted.

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**References**


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