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# SOME PROPERTIES OF $K$-SEMISTRATIFIABLE SPACES

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ABSTRACT. We study spaces admitting semistratification and $k$-semistratifications with (CF) property. The class of $k$-semistratifiable spaces with (CF) property lies between the class of Lašnev spaces and that of $k$-semistratifiable spaces, and really differs from the classes of stratifiable spaces and $\aleph$-spaces.

## 1. INTRODUCTION

All spaces are assumed to be regular Hausdorff topological spaces. The letter $\tau$ denotes the topology of a space $X$. We denote by the letter $\omega$ the set of all positive integers.

In his paper [8], Lutzer introduced the class of $k$-semistratifiable spaces, which lies between the class of stratifiable spaces in the sense of Borges [1] and Ceder [2] and the class of semistratifiable spaces introduced by Michael and studied by Creede. The class of $\sigma$-spaces introduced by Okuyama lies between that of stratifiable spaces and that of semistratifiable spaces. In this paper, we consider the limited classes of $k$-semistratifiable and semistratifiable spaces with (CF) property defined below. We give a few characterizations of Lašnev spaces in terms of $k$-semistratifiable spaces and CF families which are introduced here.

Throughout this paper, $\sigma$-spaces are spaces with a $\sigma$-discrete network or equivalently, $\sigma$-closure-preserving network, and $\aleph$-spaces are spaces with a $\sigma$-locally finite $k$-network. Stratifiable spaces are spaces with the stratification. As for the definition of stratifications, refer to Borges [1].

## 2. $K$-SEMISTRATIFIABLE SPACES WITH (CF) PROPERTY

We state the original definition of $k$-semistratifiable spaces.

**Definition 1** (Lutzer [8]). A space $X$ is called a *$k$-semistratifiable space* if there exists a function $S\colon \omega \times \tau \to$ {closed subsets of $X$} such that:

(a) For each $U \in \tau, U = \bigcup\{S(n, U)\colon n \in \omega\}$.

(b) If $U$, $V \in \tau$ and $U \subset V$, then $S(m, U) \in S(m, V)$ for each $m$.

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(c) If $C \subset U \in \tau$ with $C$ compact, then $C \subset S(m, U)$ for some $m$. (We call $S$ a *k-semistratification of* $X$.)

**Definition 2** ([10, Definition 3.1]). A family $\mathscr{U}$ of subsets of a space $X$ is called *finite on compact subsets* of $X$, briefly CF *in* $X$, if $\mathscr{U}/K$ is a finite family for any compact subsets $K$ of $X$.

**Definition 3.** A semistratification or $k$-semistratification $S$ of a space $X$ is called to *have* (CF) *property* if the following condition (CF) is satisfied:

(CF) For each $n \in \omega$, $\{S(n, U) \colon U \in \tau\}$ is CF in $X$.

A space having $S$ with (CF) property is called a *semistraifiable* or a *k-semistratifiable space with* (CF) *property*, respectively.

**Theorem 1.** *If a space $X$ has a $\sigma$-HCP (= hereditarily closure-preserving) k-network, then $X$ is a k-semistratifiable space with* (CF) *property.*

*Proof.* Let $\bigcup\{\mathscr{H}_n \colon n \in \omega\}$ be a $k$-network for $X$, where, for each $n$, $\mathscr{H}_n \subset \mathscr{H}_{n+1}$ and $\mathscr{H}_n$ is an HCP family of closed subsets of $X$. For each $(n, U) \in \omega \times \tau$, let

$$S(n, U) = \bigcup\{H \in \mathscr{H}_n \colon H \subset U\}.$$

Then it is easily seen from [10, Proposition 3.2] that $S$ is a $k$-semistratification with (CF) property.

**Example 1.** There exists a stratifiable, $k$-semistratifiable space with (CF) property, but does not have a $\sigma$-HCP $k$-network.

*Proof.* Let $Y$ be a non-metrizable Lašnev space which has no $\sigma$-locally finite $k$-network. (For example, let $Y$ be the quotient space obtained from $\bigoplus\{S_\alpha \colon \alpha < \omega_1\}$ by identifying all the limit points, where each $S_\alpha$ is the convergent sequence with its limit point.) Then by [6] the product space $X = Y \times [0, 1]$ has no $\sigma$-HCP $k$-network. $X$ is obviously a stratifiable space. By Theorem 5, stated later, $X$ is a $k$-semistratifiable space with (CF) property.

**Theorem 2.** *For a space $X$, the following are equivalent*:

(1) *$X$ is a Lašnev space.*
(2) *$X$ is a Fréchet, k-semistratifiable space with* (CF) *property.*
(3) *$X$ is a Fréchet space which has a $\sigma$-*CF *pseudobase.*

*Proof.* (1) $\to$ (2) follows from [3] and Theorem 1.
(2) $\to$ (3). Let $S$ be the $k$-semistratification of $X$ with (CF) property. Then

$$\bigcup[\{S(n, U) \colon U \in \tau\} \colon n \in \omega\}]$$

is a $\sigma$-CF pseudobase of $X$.
(3) $\to$ (1) follows from [10, Theorem 4.1, (9)].

**Corollary.** *A space $X$ is metrizable if and only if $X$ is a first countable, k-semistratifiable space with* (CF) *property.*

We notice that a Lašnev space cannot be characterized to be a Fréchet space with a $\sigma$-HCP "pseudobase" [5]. For the next example, we prepare a lemma.

**Lemma.** *If a space $X$ has a $\sigma$-HCP $k$-network $\mathscr{H}$ of closed subsets of $X$, then $X = X_1 \cup X_2$, where $X_1$ is a $\sigma$-discrete closed subspace and $X_2$ is an $\aleph$-space such that for each $p \in X_2$, $\mathscr{H}$ is $\sigma$-locally finite at $p$ in $X$.*

*Proof.* Let $\mathscr{H} = \bigcup\{\mathscr{H}_n : n \in \omega\}$, where for each $n$ $\mathscr{H}_n \subset \mathscr{H}_{n+1}$ and $\mathscr{H}_n$ is an HCP family of closed subsets of $X$. Let

$$X_1 = \{p \in X : \bigcap\{H \in \mathscr{H}_n : p \in H\} \text{ is a finite subset for some } n\}.$$

Then by the same argument as in [11, Theorem 3.6], we can show that $X_1$ is a countable union of discrete closed subsets of $X$ and $X_2 = X - X_1$ has the required property.

**Example 2.** There exists a stratifiable space which has no $\sigma$-HCP $k$-network.

*Proof.* For each $\alpha < \omega_1$, let $T_\alpha$ be the copy of the subspace $T = \{(x, y) : 0 \leq x, y \leq 1\}$ of $\mathbf{R}^2$ and $f_\alpha : T \to T_\alpha$ its homeomorphism. Let $X$ be the quotient space obtained from $\bigoplus\{T_\alpha : \alpha < \omega_1\}$ by identifying $\{f_\alpha((x, 0) : \alpha < \omega_1\}$ for each $x$ with $0 \leq x \leq 1$. Since $X$ is dominated by metric spaces, $X$ is a stratifiable space [1, Theorem 7.2]. If $X$ has a $\sigma$-HCP $k$-network $\mathscr{H}$, then by the above, there exists a point $p = f(f_\alpha((x, 0))) \in X$ such that $\mathscr{H}$ is $\sigma$-locally finite at $p$ in $X$, where $f : \bigoplus T_\alpha \to X$ is the quotient mapping. But, by [7, Remark 2] this is a contradiction.

**Example 3.** There exists a $k$-semistratifiable space which does not have (CF) property.

*Proof.* Let $X$ be the same space as in [2, Example 9.2]. Then $X$ is a first countable, non-metrizable stratifiable space. By the Corollary to Theorem 2, $X$ has no $k$-semistratification with (CF) property.

From the argument as in Theorem 1, the following is easily seen.

**Theorem 3.** *Any $\sigma$-space is a semistratifiable space with* (CF) *property.*

The converse is not known. However, we have a partial answer to it.

**Theorem 4.** *If a space $X$ is a Fréchet, semistratifiable space with* (CF) *property, then $X$ is a $\sigma$-space.*

*Proof.* It is easy to see that $X$ has a $\sigma$-CF network $\mathscr{H}$. By [10, Proposition 3.3], $\mathscr{H}$ is a $\sigma$-closure-preserving network. Hence $X$ is a $\sigma$-space.

The following example shows that any semistratifiable space need not have (CF) property.

**Example 4.** There exists a first countable, semistratifiable space which is not a $\sigma$-space.

*Proof.* Let $X$ be the space in [4, Example 9.10]. Since $X$ is not a $\sigma$-space, by Theorem 4, $X$ is not a semistratifiable space with (CF) property.

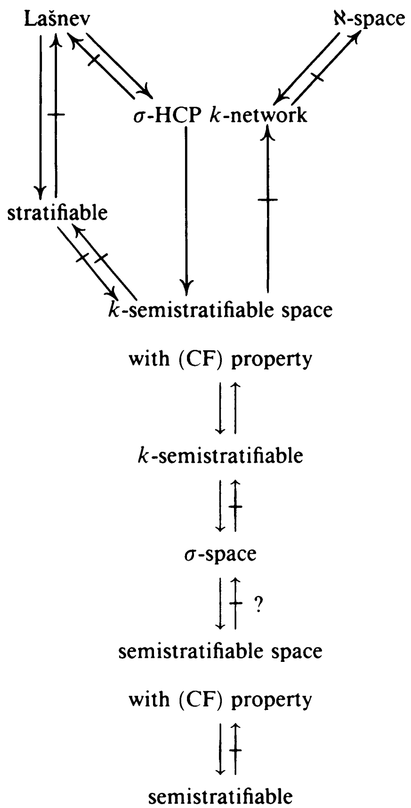
**Theorem 5.** *If a space $X$ is embedded into a countable product of Lašnev spaces, then $X$ is a $k$-semistratifiable space with* (CF) *property.*

*Proof.* By the same method as in [9, Lemma 5.1 and Proposition 6.1] and by [10, Proposition 3.3], we can show that $X$ has a $\sigma$-closure-preserving, CF family $\bigcup_n \mathscr{H}_n$ of closed subsets of $X$, which forms a $k$-network for $X$. For each $(n, U) \in \omega \times \tau$, let

$$S(n, U) = \bigcup \left\{ H \in \bigcup_{t \le n} \mathscr{H}_t : H \subset U \right\}.$$

Then $S$ is a $k$-semistratification with (CF) property.

The other known implications are indicated by the diagram below. The remaining proofs are easy and well-known, and therefore they are omitted.

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