

A REMARK ON THE GROMOV CONVERGENCE THEOREM

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ABSTRACT. In [3] M. Gromov introduced the concept of convergence of Riemannian manifolds and he proved the convergence theorem. Since that time the theorem has been developed in detail (see [5], [7], [2]), and we know that it contains some interesting applications. Nevertheless there seems to be an inadequate way of applying the convergence theorem. The purpose of this paper is to present an example which shows that it is not correct.

1. INTRODUCTION

First of all we recall the convergence theorem. Let M be an n -dimensional compact Riemannian manifold. We use the notation K_M for the sectional curvature of M , $\text{vol}(M)$ for the volume and $\text{diam}(M)$ for the diameter. Let $\mathcal{M} = \mathcal{M}(n, \Delta, D, v)$ denote the class of all n -dimensional Riemannian manifolds M satisfying $|K_M| \leq \Delta$, $\text{diam}(M) \leq D$, $\text{vol}(M) \geq v > 0$.

Theorem 1.1 (Katsuda [5], Peters [7]). *The Hausdorff topology coincides with the Lipschitz topology on \mathcal{M} .*

Furthermore, we have:

Theorem 1.2 (Peters [7], Greene-Wu [2]). *Given $0 < \alpha < 1$, let $\{(M_i, g_i)\}_{i=1,2,\dots}$ be a sequence of Riemannian manifolds in \mathcal{M} . Then there exist a $C^{1,\alpha}$ -Riemannian manifold (M, g_∞) and a subsequence $\{(M_{i_j}, g_{i_j})\}_{j=1,2,\dots}$ which converges to (M, g_∞) with respect to Lipschitz topology.*

For simplicity we denote the subsequence $\{i_j\}_{j=1,2,\dots}$ by $\{j\}$. In the above case, by definition, there exist diffeomorphisms $f_j: M \rightarrow M_j$ such that $\{g'_j := f_j^* g_j\}$ converges to g_∞ on M . Let $p \in M$. The metric g'_j induces the exponential map $\exp_p^{g'_j}: T_p M \rightarrow M$. By Rauch's comparison theorem $\exp_p^{g_\infty}$ is Lipschitz continuous. Hence $\exp_p^{g_\infty}$ is differentiable almost everywhere. Thus it is reasonable to expect the following (see [1], [4, 5.15]) proposition.

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Proposition 1.3. *At any point in the tangent space $T_p M$ where $d \exp_p^{g_\infty}$ makes sense*

$$\lim_{j \rightarrow \infty} d \exp_p^{g_j'} = d \exp_p^{g_\infty}.$$

However, Proposition 1.3 is false. The proof of a diameter pinching theorem ([1]) contains the same mistake (for another proof of the theorem, see [5], also [8]). In this paper we present a simple counterexample to the proposition.

2. A COUNTEREXAMPLE

Let $M_\infty = (M, g_\infty)$ be a 2-dimensional compact Riemannian manifold with injectivity radius at least 2 and let p be a point of M . We construct a sequence of compact Riemannian manifolds $M_i = (M, g_i)$ from (M, g_∞) as follows. Let $\varphi: B_p(2) \setminus \{p\} \rightarrow (0, 2) \times S^1$ be the polar coordinate system centered at p . Assume that g_∞ is written as

$$(\varphi^{-1})^* g_\infty = dr^2 + G_\infty^2(r) d\theta^2$$

at $(r, \theta) \in (0, 2) \times S^1$ for some C^∞ -function G_∞ . Let $\lambda: \mathbf{R} \rightarrow [0, 1/2]$ denote a nonnegative C^∞ -function such that

$$\begin{aligned} \lambda(r) &= 0 && \text{if } r \leq \frac{1}{4}, 1 \leq r, \\ \lambda(r) &= 1/2 && \text{if } \frac{1}{2} \leq r \leq \frac{3}{4}. \end{aligned}$$

For any integer i the metric g_i on $B_p(2) \setminus \{p\}$ is defined to be

$$(\varphi^{-1})^* g_i = dr^2 + G_i^2(r, \theta) d\theta^2,$$

where $G_i(r, \theta) := G_\infty(r)(1 + \lambda(r) \cos i\theta)$. Since $G_i = G_\infty$ for $r \leq 1/4$ and $1 \leq r$, we can extend g_i on M so that $g_i = g_\infty$ on $B_p(1/4)$ and $M \setminus B_p(1)$.

We show that the sequence $\{M_i\}_{i=1,2,\dots}$ obtained as above is in $\mathcal{M}(2, \Delta, D, v)$ for some constants Δ, D, v . It is sufficient to consider $g_i|_{B_p(1)}$. The sectional curvature K_i of $g_i|_{B_p(1)}$ is

$$K_i = -\frac{(G_i)_{rr}}{G_i} = -\frac{(G_\infty)_{rr}}{G_\infty} - \frac{2(G_\infty)_r \lambda' + G_\infty \lambda''}{G_\infty} \cdot \frac{\cos i\theta}{1 + \lambda \cos i\theta}.$$

Thus K_i is uniformly bounded for i . The diameter of $B_p(1)$ with respect to g_i is at most 2. And the volume of $B_p(1)$ with respect to g_i is that of $B_p(1)$ with respect to g_∞ .

Claim 2.1. $d_L(M_i, M_\infty) \rightarrow 0$ as $i \rightarrow \infty$.

Proof. To estimate the distance between M_i and M_∞ we define the diffeomorphism $F_i: M_i \rightarrow M_\infty$ by

$$F_i|(M \setminus B_p(1)) = id_{M \setminus B_p(1)}$$

and

$$\varphi \circ (F_i|_{B_p(1)}) \circ \varphi^{-1}(r, \theta) = \left(r, \theta + \frac{\lambda(r)}{i} \sin i\theta \right).$$

Let q be a point in $B_p(1)$ and u be a unit tangent vector at q , then

$$d\varphi_q u = \cos \phi \frac{\partial}{\partial r} + \frac{\sin \phi}{G_i(r)} \frac{\partial}{\partial \theta}$$

for some ϕ , where $\varphi(q) = (r, \theta)$. Then

$$g_\infty(dF_i u, dF_i u) = \cos^2 \phi + G_\infty^2 \cdot \left(\frac{\lambda'}{i} \sin i\theta \cos \phi + \frac{\sin \phi}{G_\infty} \right)^2.$$

Therefore there exists a constant $c \geq 0$ which is independent of i such that for any u in the unit tangent bundle of M_i ,

$$1 - \frac{c}{i} \leq |dF_i u| \leq 1 + \frac{c}{i}.$$

Similarly, we find a constant $c' \geq 0$ which is independent of i such that for any w in the unit tangent bundle of M_∞ ,

$$1 - \frac{c'}{i} \leq |dF_i^{-1} w| \leq 1 + \frac{c'}{i}.$$

This completes the proof.

Let $g'_i := (F_i^{-1})^* g_i$. We can easily show that $\{d \exp_p^{g'_i}\}$ does not converge for almost all $u \in T_p M$ with $1/4 \leq |u| \leq 1$: Since $F_i(p) = p$ and $dF_i: T_p M_i \rightarrow T_p M_\infty$ is an isometry, $T_p M_i$ and $T_p M_\infty$ are naturally identified as metric vector spaces. Since $\partial/\partial r$ and $1/G_i \partial/\partial \theta$ make an orthonormal base,

$$\begin{aligned} \det d(\exp_p^{g'_i})_u &= \det dF_i \circ d(\exp_p^{g_i})_u \\ &= (1 + \lambda \cos i\theta) \times \frac{G_i}{r} \\ &= (1 + \lambda \cos i\theta)^2 \times \frac{G_\infty}{r}. \end{aligned}$$

Similarly

$$\det d(\exp_p^{g_\infty})_u = \frac{G_\infty}{r}.$$

Hence $\{\det d(\exp_p^{g'_i})_u\}$ does not converge to $\det d(\exp_p^{g_\infty})_u$ for almost all $u \in T_p M$ with $1/4 \leq |u| \leq 1$.

Let $\{f_i\}$ be a sequence of diffeomorphisms $f_i: M_\infty \rightarrow M_i, i = 1, 2, \dots$. We say that $\{d \exp^i\}$ converges to $d(\exp_p^{g_\infty})$ at p with respect to $\{f_i\}$ if $\{d(\exp_p^{g'_i})_u: T_p M \rightarrow TM\}$ converges to $d(\exp_p^{g_\infty})_u$ for almost all $u \in T_p M$ for some choice of subsequence $\{f_j\}$ such that $\{g'_j := f_j^* g_j\}$ is convergent. We proved that $\{d \exp^i\}$ does not converge to $d(\exp_p^{g_\infty})$ with respect to $\{F_i^{-1}\}$. However $\{F_i^{-1}\}$ is not $C^{2,\alpha}$ -bounded on M_∞ . We shall prove in the following that there exists a sequence of $C^{2,\alpha}$ -bounded diffeomorphisms which is constructed from $\{F_i^{-1}\}$ by the center of mass technique as in the proof of Theorem 1.2 and with respect to which $\{d \exp^i\}$ does not converge to $d(\exp_p^{g_\infty})$.

Assume that $0 < \delta \ll R \ll 1$. If $\{p_\alpha\}$ is a δ -maximal system on M_∞ , then $\{F_i^{-1}(p_\alpha)\}$ is an almost δ -net on M_i . A family of local embeddings $\phi_\alpha^i: H(p_\alpha; R) \rightarrow M_i$ is constructed by using harmonic R -balls $H(p_\alpha; R)$. In our case, if $p_\alpha \in B(p; 1/4 - R)$, we may consider ϕ_α^i to be the identity on $H(p_\alpha; R)$. Hence the maps $\{f_j\}$ constructed from $\{\phi_\alpha^i\}$ by the center of mass have the property that f_j is identical on a neighborhood of p . By taking a subsequence $\{f_j\}$ we may assume $\{f_j\}$ converges to id_M with respect to the $C^{2,\alpha}$ -norm on M and $\{g'_j := f_j^* g_j\}$ converges to g_∞ with respect to the $C^{1,\alpha}$ -norm. Since

$$\det df_j \circ d(\exp_p^{g'_j})_u = \det d(\exp_p^{g_j})_u,$$

if we suppose that $\lim_{j \rightarrow \infty} d(\exp_p^{g'_j})_u = d(\exp_p^{g_\infty})_u$, then

$$\lim_{j \rightarrow \infty} \det d(\exp_p^{g_j})_u = \det d(\exp_p^{g_\infty})_u.$$

However this is impossible for almost all $u \in T_p M$ with $1/4 < |u| < 1$ as above.

Remark 2.2. If we take the unit sphere $S^2(1)$ as M_∞ and change λ properly, we have an example such that every M_i has positive sectional curvature.

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