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A CHARACTERIZATION OF SMOOTH CANTOR BOUQUETS

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ABSTRACT. We prove that all smooth fans having a dense set of endpoints are topologically equivalent.

Let X be a smooth fan whose set of endpoints is dense in X. Such fans have been constructed, e.g., by J. H. Roberts [6], who proved that the space of rational sequences of the Hilbert cube can be embedded in the Cantor fan, and by A. Lelek [3], who showed the existence of a fan whose (one-dimensional) set of endpoints can be connectified by adding the vertex. Lately, spaces similar to $X \setminus \{v\}$, where v is the vertex of X, were discovered to be Julia sets of some nice analytic functions (see R. L. Devaney and M. Krych [2]; see also J. C. Mayer [4]). We are going to prove that all such examples are homeomorphic.

Theorem. *All smooth fans having dense set of endpoints are topologically equivalent.*

Let us recall that a continuum X is said to be *hereditarily unicoherent* if $K \cap L$ is connected for every pair K, L of subcontinua of X. A continuum X is called a *dendroid* if it is arcwise connected and hereditarily unicoherent. By a *fan* we will mean a dendroid having exactly one ramification point; we will call this point the *vertex* of X. A fan X is said to be *smooth* if the sequence of arcs $[v, x_n]$ converges to the arc $[v, x]$ for every sequence x_n converging to x, where x, $x_n \in X$ and v is the vertex of X. If X is a fan, then $E(X)$ will denote the set of endpoints of X. If x, $y \in \mathfrak{R} \times \mathfrak{R}$, then by $|x - y|$ we will denote the Euclidean distance between points x and y and by $[x, y]$ we will mean the linear segment with endpoints x and y.

MAPPINGS BETWEEN INVERSE SYSTEMS

The following lemma is similar to [5, Theorem 2$'$]. The proof is a standard inductive argument and is left to the reader.

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Lemma 1. *Suppose* P_n *and* Q_n *are compact metric spaces,* $X = \varprojlim \langle P_n, p_m^n \rangle$, $Q = \varprojlim \langle Q_n, q_m^n \rangle$, *and* $\mathscr{F}_n$ *is a class of mappings from* P_n *onto* Q_n *such that for every* n, *positive real* ε, *and mapping* $f \in \mathscr{F}_n$, *there exists a mapping* $g \in \mathscr{F}_{n+1}$ *such that the following diagram is* ε*-commutative, i.e.* $\operatorname{dist}(g \circ p_n^{n+1}(x), q_n^{n+1} \circ g(x)) < \varepsilon$ *for each* $x \in P_{n+1}$.

$$\begin{array}{ccc} P_n & \xleftarrow{\;p_n^{n+1}\;} & P_{n+1} \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ Q_n & \xleftarrow{\;q_n^{n+1}\;} & Q_{n+1} \end{array}$$

Let ε_n *be a decreasing sequence converging to* 0. *Then for each* n *there exists an* $h_n \in \mathscr{F}_n$ *such that the following diagram is* ε_n*-commutative for every* $k \le n \le m$.

$$(*) \qquad \begin{array}{ccccc} & & P_n & \xleftarrow{\;p_n^m\;} & P_m \\ & & \big\downarrow{\scriptstyle h_n} & & \big\downarrow{\scriptstyle h_m} \\ Q_k & \xleftarrow{\;q_k^n\;} & Q_n & \xleftarrow{\;q_n^m\;} & Q_m \end{array}$$

By Mioduszewski's result [5, Theorem 2], the sequence h_n induces a map $h: X \xrightarrow{\text{onto}} Y$ defined by $h((x_1, x_2, \ldots)) = (y_1, y_2, \ldots)$, where

$$y_n = \lim_{m \to \infty} q_n^m \circ h_m(x_n).$$

Let us omit the standard proof of the following lemma.

Lemma 2. *Let* $X = \varprojlim \langle P_n, p_m^n \rangle$, $Y = \varprojlim \langle Q_n, q_m^n \rangle$, $\{\varepsilon_n : n = 1, 2, \ldots\}$, $\{h_n : n = 1, 2, \ldots\}$, *and* h *be as in the statement of Lemma* 1. *Assume, in addition, that spaces* X, P_n, *and* Q_n *are embedded in the Hilbert cube in such a way that* $p_n : X \to P_n$ *and* $q_n : Y \to Q_n$ *are* $1/n$*-mappings. Let* $x_n \in P_n$ *be a sequence converging to* $x \in X$. *Then* $\lim_{n \to \infty} h_n(x_n) = h(x)$.

Preliminary lemmas

Let C denote a Cantor set lying in $[0,1] \times \{1\} \subseteq \mathfrak{R} \times \mathfrak{R}$. Let $v = \langle 1/2, 0 \rangle$ and let T be the Cantor fan being the union of all linear segments joining v with points of C. Let $\rho: T \to [0,1]$ be the natural second coordinate projection. Let X be a fan such that $E(X)$ is dense in X. By the result of Carruth [1], we may assume that X is embedded in T. There is a natural monotone mapping $\pi: X \setminus \{v\} \to C$ such that $e \in [v, \pi(e)]$ for every $e \in E(X)$. We may assume that $\pi(E(X))$ is dense in C. The assertion of the following lemma is a consequence of the density of $E(X)$.

Lemma 3. *For every point $e \in E(X)$ there is a sequence $\{e_n \in E(X)\colon n = 1, 2, \ldots\}$ such that $\pi(e) = \lim_{n\to\infty} \pi(e_n)$ and $\operatorname{cl}\{e_n\colon n = 1, 2, \ldots\} = [v, e] \cup \{e_n\colon n = 1, 2, \ldots\}$.*

For a subset A of C define

$$h(A) = \sup\{\rho(e)\colon e \in E(X) \text{ and } \pi(e) \in A\}.$$

Without loss of generality we may assume that $h(C) = 1$. Let us omit the easy proof of the following lemma.

Lemma 4. *For every nonempty closed-and-open subset U of C there is a point $e \in E(X)$ such that $\pi(e) \in U$ and $\rho(e) = h(U)$.*

Lemma 5. *For every $\varepsilon > 0$, nonempty closed-and-open subset U of C and point $e \in E(X)$ such that $\pi(e) \in U$ and $\rho(e) = h(U)$, there is a null partition $\{U_n\colon n = 1, 2, \ldots\}$ of $U \setminus \{\pi(e)\}$ into closed-and-open subsets of C of diameter less than ε such that $\{h(U_n)\colon n = 1, 2, \ldots\}$ is dense in $\rho([v, e])$. Moreover, if $e_n \in E(X)$ is such that $\pi(e_n) \in U_n$ and $\rho(e_n) = h(U_n)$, then $\operatorname{cl}\{e_n\colon n = 1, 2, \ldots\} = [v, e] \cup \{e_n\colon n = 1, 2, \ldots\}$.*

Proof. By Lemma 3, there is a sequence $f_m \in E(X)$ such that $\pi(e) = \lim_{m\to\infty} \pi(f_m)$ and $\operatorname{cl}\{f_m\colon m = 1, 2, \ldots\} = [v, e] \cup \{f_m\colon m = 1, 2, \ldots\}$. Let $\mathscr{T} = \{W_m\colon m = 1, 2, \ldots\}$ be a sequence of disjoint closed-and-open subsets of $U \setminus \{\pi(e)\}$ such that $\pi(f_m) \in W_m$, $\operatorname{diam} W_m < \varepsilon$, $\lim_{m\to\infty} \operatorname{diam} W_m = 0$, and $|\rho(f_m) - h(W_m)| < 1/m$. Observe that $\{h(W_m)\colon m = 1, 2, \ldots\}$ is dense in $\rho([v, e])$. Hence, any completion of $\mathscr{T}$ to a null partition of $U \setminus \{\pi(e)\}$ with mesh less than ε will satisfy the assertion of the lemma.

Construction of an inverse sequence $\langle T_n, p_m^n \rangle$ associated with X

Choose $e_0 \in E(X)$ such that $\rho(e_0) = h(C)$. Let $T_0 = [v, e_0]$ and let $p_0\colon X \to T_0$ be the horizontal projection onto T_0. Since $\rho(e_0) = h(C)$, the map p_0 is well defined. We will call e_0 the endpoint of T_0 and will write $E(T_0) = \{e_0\}$. Put $\mathscr{T}_0 = \{C\}$ and let $\mathscr{T}_1 = \{U_n\colon n = 1, 2, \ldots\}$ be a partition of $C \setminus \{\pi(e_0)\}$ guaranteed by Lemma 5 for $e = e_0$ and $\varepsilon = 1$. For every n choose $e_n \in E(X)$ such that $\pi(e_n) \in U_n$ and $\rho(e_n) = h(U_n)$. Let $T_1 = T_0 \cup \bigcup\{[v, e_n]\colon n = 1, 2, \ldots\}$. We may define $p_1\colon X \to T_1$ in such a way that $p_1|T_0$ is the identity and $p_1|\pi^{-1}(U_n)$ is the horizontal projection into $[v, e_n]$. Suppose we have already defined sets $T_0, \ldots, T_n \subset X$, mappings $p_k\colon X \to T_k$ for $k = 0, \ldots, n$, and collections $\mathscr{T}_0, \ldots, \mathscr{T}_n$ such that

(1) T_k is a fan for $k = 1, 2, \ldots, n$,
(2) $T_0 \subset T_1 \subset \cdots \subset T_n$,
(3) $E(T_0) \subset E(T_1) \subset \cdots \subset E(T_n)$,
(4) $p_{k+1}|T_k$ is the identity for $k = 0, \ldots, n-1$,
(5) $\mathscr{T}_{k+1}$ is a null family of disjoint closed-and-open sets of diameter less than or equal to $1/(k+1)$, refining $\mathscr{T}_k$ and such that $\bigcup \mathscr{T}_{k+1} = C \setminus \pi(E(T_k))$,

(6) for every $U \in \mathscr{T}_k$ there is a unique point $e_k(U) \in E(T_k)$ such that $\pi(e_k(U)) \in U$; further,
(a) $\rho(e_k(U)) = h(U)$,
(b) $p_k|\pi^{-1}(U)$ is the horizontal projection into $[v, e_k(U)]$ and
(c) if $W \in \mathscr{T}_{k-1}$ and $\mathscr{T} = \{U \in \mathscr{T}_k : U \subset W\}$, then

$$\mathrm{cl}\{e_k(U) : U \in \mathscr{T}\} = \{e_k(U) : U \in \mathscr{T}\} \cup [v, e_{k-1}(W)].$$

Observe that in view of Lemmas 4 and 5 the induction can be continued. Define $p_m^n = p_m|T_n$ for $m \leq n$ and observe that $p_m = p_m^n \circ p_n$ and $\rho(x) = \rho \circ p_n(x)$ for every n and $x \in X$. Since $X = \mathrm{cl}\bigcup\{T_n : n = 0, 1, \ldots\}$ and every $p_n : X \to T_n$ is a $1/n$-map, the space X is homeomorphic to $\varprojlim \langle T_n, p_m^n\rangle$ and the maps p_n converge to the identity on X. We will say that the above inverse sequence is *associated with the fan* X.

Construction of a homeomorphism

Now, let X and Y be smooth fans having a dense set of endpoints and let $\langle P_n, p_m^n\rangle$ and $\langle Q_n, q_m^n\rangle$ be inverse sequences associated with X and Y, respectively. To complete the proof of the theorem we will construct a sequence of homeomorphisms $h_n : P_n \overset{onto}{\to} Q_n$ inducing a homeomorphism between the limit spaces.

Let v be the vertex of X and w the vertex of Y. We may assume that $\rho(X) = \rho(Y) = [0, 1]$. Let $h_0 : P_0 \overset{onto}{\to} Q_0$ be the linear map such that $h_0(v) = w$. Let $\mathscr{F}_0 = \{h_0\}$. Let $\{e_n : n = 1, 2, \ldots\} = E(P_1) \setminus E(P_0)$ and $\{f_n : n = 1, 2, \ldots\} = E(Q_1) \setminus E(Q_0)$. We may find a permutation φ of positive integers such that

$$|\rho(e_n) - \rho(f_{\varphi(n)})| \leq \min\left\{\frac{1}{\alpha(n)\sqrt{5}}, \frac{\rho(e_n)}{4}\right\}$$

for every n, where $\alpha(n) = \min\{n, \varphi(n)\}$.

Let $h_1 : P_1 \overset{onto}{\to} Q_1$ be the extension of h_0, such that h_1 maps $[v, e_n]$ linearly onto $[w, f_{\varphi(n)}]$ for every n. Observe that if $x \in P_1 \setminus \{v\}$, then $\pi \circ q_0^1 \circ h_1(x) = \pi \circ h_0 \circ p_0^1(x)$. Hence, $|q_0^1 \circ h_1(z) - h_0 \circ p_0^1(z)| \leq 1/2$ for every z.

For each nonnegative integer n we will inductively construct a class $\mathscr{F}_n$ of homeomorphisms mapping P_n onto Q_n such that for every $h \in \mathscr{F}_n$ and $\varepsilon > 0$ there is a $g \in \mathscr{F}_{n+1}$ satisfying the following conditions:

(7) $g|P_n = h$,
(8) for every $e \in E(P_n)$ the function $h|[v, e] : [v, e] \overset{onto}{\to} [w, h(e)]$ is a linear homeomorphism,
(9) $\pi \circ q_n^{n+1} \circ g(x) = \pi \circ h \circ p_n^{n+1}(x)$,
(10) $|\rho \circ q_n^{n+1} \circ q(x) - \rho \circ h \circ p_n^{n+1}(x)| < \varepsilon$, and
(11) $|\rho(x) - \rho \circ h(x)| < \rho(x)/4$ for each $x \in P_n$.

Suppose that classes of homeomorphisms $\mathscr{F}_i$ satisfying (7)–(11) have already been defined for each i, $0 \leq i \leq n$. Let $\mathscr{U}_i[\mathscr{V}_i]$ be the partition of $C \setminus E(P_{i-1})$ [of $C \setminus E(Q_{i-1})$, respectively] used for the construction of $P_i[Q_i]$, where $i = 1, 2, \ldots, n$. Let $\{U_k : k = 1, 2, \ldots\}$ be an enumeration of $\mathscr{U}_n$ and let $e_k = e(U_k)$. The homeomorphism $h: P_n \to Q_n$ maps each e_k to a point $f_k \in E(Q_n)$. Let V_k be the unique element in $\mathscr{V}_n$ containing f_k. Then $\{V_k : k = 1, 2, \ldots\}$ is an enumeration of $\mathscr{V}_n$. For every k, let $\mathscr{U}_{n+1}(U_k) = \{U \in \mathscr{U}_{n+1} : U \subset U_k\}$, and $\mathscr{V}_{n+1}(V_k) = \{V \in \mathscr{V}_{n+1} : V \subset V_k\}$. Let $\{U_{k,j} : j = 1, 2, \ldots\}$ and $\{V_{k,j} : j = 1, 2, \ldots\}$ be enumerations of $\mathscr{U}_{n+1}(U_k)$ and $\mathscr{V}_{n+1}(V_k)$, respectively. Put $e_{k,j} = e(U_{k,j})$ and $f_{k,j} = f(V_{k,j})$. Recall that $\rho(x) = \rho \circ p_m^n(x)$. By (11), $|\rho(e_{k,j}) - \rho \circ h \circ p_n^{n+1}(e_{k,j})| < \rho(e_{k,j})/4$.

For every k we may find a permutation φ_k of positive integers such that

$$(**) \qquad |\rho(e_{k,j}) - \rho(f_{k,\varphi_k(j)})| < \frac{\rho(e_{k,j})}{4}$$

and

$$(***) \qquad |\rho \circ h \circ p_n^{n+1}(e_{k,j}) - \rho(f_{k,\varphi_k(j)})| < \min\left\{\varepsilon, \frac{1}{\alpha(j) + k}\right\},$$

where $\alpha(j) = \min\{j, \varphi_k(j)\}$. Let g be the extension of h which maps $[v, e_{k,j}]$ linearly onto $[w, f_{k,\varphi_k(j)}]$. Then (7) and (8) follow immediately; (10) and the continuity of g follow from $(***)$ and the linearity of g; and (11) for g follows from (11) for h, the condition $(**)$, and the fact that $h|[v, e_{k,j}]$ is linear.

Let $\mathscr{F}_{n+1}$ be the class of homeomorphisms g mapping P_{n+1} onto Q_{n+1} obtained as described above for every $h \in \mathscr{F}_n$ and $\varepsilon = 1/r$, where $r = 1, 2, \ldots$.

By Lemma 1, we can select a sequence $h_n \in \mathscr{F}_n$ of homeomorphisms such that for each $k \leq n \leq m$ diagram $(*)$ is $1/2^n$-commutative. Hence the homeomorphisms h_n induce a continuous map $h: X \overset{onto}{\to} Y$ defined by $h(x) = y$, where $q_s(y) = \lim_{n\to\infty} q_s^n \circ h_n \circ p_n(x)$. To complete the proof it suffices to show that h is one-to-one.

Let $x \neq a \in X$ and let $x_n = p_n(x)$, $a_n = p_n(a)$, $h(x) = y$, and $h(a) = b$. Suppose first that for some n, $\pi(x_n) \neq \pi(a_n)$. By (9), the definition of h and the fact that each h_n is a homeomorphism, $\pi \circ q_n \circ h(x) \neq \pi \circ q_n \circ h(a)$, and $h(x) \neq h(a)$. Hence we may assume that either $\pi(x_n) = \pi(a_n)$, for each n or $a = v$.

Then there exists a (unique) $e \in E(P_n)$ such that $p_n([v, e]) \subset [v, e_n]$. Since $e \in E(X)$, $\mathrm{Lim}_{n\to\infty}[v, e_n] = [v, e]$ and $\lim_{n\to\infty} \rho(e_n) = \rho(e)$. Clearly, $\rho(e_{n+1}) \leq \rho(e_n)$ for each n. By (11),

$$\rho \circ h_n(e_n) > \frac{3}{4}\rho(e_n) \geq \frac{3}{4}\rho(e)$$

for each n. Since each h_n is linear,

$$|\rho \circ h_n(x_n) - \rho \circ h_n(y_n)| \geq \frac{3}{4}|\rho(x_n) - \rho(y_n)|$$

for each n and $x_n, y_n \in [v, e_n]$. Since $\mathrm{Lim}_{n\to\infty}[v, e_n] = [v, e]$, there exist c_n, $d_n \in [v, e_n]$ such that $\lim_{n\to\infty} c_n = x$ and $\lim_{n\to\infty} d_n = a$. Hence,

$$\lim_{n\to\infty} |\rho \circ h_n(c_n) - \rho \circ h_n(d_n)| \geq \lim_{n\to\infty} |\rho(c_n) - \rho(d_n)| = |\rho(x) - \rho(a)| > 0.$$

Hence, by Lemma 2, $|\rho \circ h(x) - \rho \circ h(a)| > 0$ and $h(x) \neq h(a)$.

Added in proof. The main result of this paper was proved independently by W. J. Charatonik [*The Lelek fan is unique*, (to appear in *Houston J. of Math.*)].

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