A CHARACTERIZATION OF SMOOTH CANTOR BOUQUETS

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(Communicated by James E. West)

Abstract. We prove that all smooth fans having a dense set of endpoints are topologically equivalent.

Let $X$ be a smooth fan whose set of endpoints is dense in $X$. Such fans have been constructed, e.g., by J. H. Roberts [6], who proved that the space of rational sequences of the Hilbert cube can be embedded in the Cantor fan, and by A. Lelek [3], who showed the existence of a fan whose (one-dimensional) set of endpoints can be connectified by adding the vertex. Lately, spaces similar to $X \setminus \{v\}$, where $v$ is the vertex of $X$, were discovered to be Julia sets of some nice analytic functions (see R. L. Devaney and M. Krych [2]; see also J. C. Mayer [4]). We are going to prove that all such examples are homeomorphic.

Theorem. All smooth fans having dense set of endpoints are topologically equivalent.

Let us recall that a continuum $X$ is said to be hereditarily unicoherent if $K \cap L$ is connected for every pair $K, L$ of subcontinua of $X$. A continuum $X$ is called a dendroid if it is arcwise connected and hereditarily unicoherent. By a fan we will mean a dendroid having exactly one ramification point; we will call this point the vertex of $X$. A fan $X$ is said to be smooth if the sequence of arcs $[v, x_n]$ converges to the arc $[v, x]$ for every sequence $x_n$ converging to $x$, where $x_n \in X$ and $v$ is the vertex of $X$. If $X$ is a fan, then $E(X)$ will denote the set of endpoints of $X$. If $x, y \in \mathbb{R} \times \mathbb{R}$, then by $|x - y|$ we will denote the Euclidean distance between points $x$ and $y$ and by $[x, y]$ we will mean the linear segment with endpoints $x$ and $y$.

Mappings between inverse systems

The following lemma is similar to [5, Theorem 2']. The proof is a standard inductive argument and is left to the reader.
Lemma 1. Suppose $P_n$ and $Q_n$ are compact metric spaces, $X = \lim (P_n, p_n^m)$, $Q = \lim (Q_n, q_n^m)$, and $\mathcal{F}_n$ is a class of mappings from $P_n$ onto $Q_n$ such that for every $n$, positive real $\varepsilon$, and mapping $f \in \mathcal{F}_n$, there exists a mapping $g \in \mathcal{F}_{n+1}$ such that the following diagram is $\varepsilon$-commutative, i.e. $\text{dist}(g \circ p_{n+1}^m(x), q_{n+1}^m \circ g(x)) < \varepsilon$ for each $x \in P_{n+1}$.

\[
\begin{array}{ccc}
P_n & \xrightarrow{p_{n+1}^m} & P_{n+1} \\
\downarrow f & & \downarrow g \\
Q_n & \xleftarrow{q_{n+1}^m} & Q_{n+1}
\end{array}
\]

Let $\varepsilon_n$ be a decreasing sequence converging to 0. Then for each $n$ there exists an $h_n \in \mathcal{F}_n$ such that the following diagram is $\varepsilon_n$-commutative for every $k \leq n \leq m$.

\[
\begin{array}{ccc}
P_n & \xrightarrow{p_n^m} & P_m \\
\downarrow h_n & & \downarrow h_m \\
Q_k & \xleftarrow{q_n^m} & Q_m
\end{array}
\]

By Mioduszewski's result [5, Theorem 2], the sequence $h_n$ induces a map $h: X \to Y$ defined by $h((x_1, x_2, \ldots)) = (y_1, y_2, \ldots)$, where

\[y_n = \lim_{m \to \infty} q_n^m \circ h_m(x_n)\]

Let us omit the standard proof of the following lemma.

Lemma 2. Let $X = \lim (P_n, p_n^m)$, $Y = \lim (Q_n, q_n^m)$, $\{e_n: n = 1, 2, \ldots\}$, $\{h_n: n = 1, 2, \ldots\}$, and $h$ be as in the statement of Lemma 1. Assume, in addition, that spaces $X$, $P_n$, and $Q_n$ are embedded in the Hilbert cube in such a way that $p_n: X \to P_n$ and $q_n: Y \to Q_n$ are $1/n$-mappings. Let $x_n \in P_n$ be a sequence converging to $x \in X$. Then $\lim_{n \to \infty} h_n(x_n) = h(x)$.

Preliminary lemmas

Let $C$ denote a Cantor set lying in $[0, 1] \times \{1\} \subseteq \mathbb{R} \times \mathbb{R}$. Let $v = (1/2, 0)$ and let $T$ be the Cantor fan being the union of all linear segments joining $v$ with points of $C$. Let $\rho: T \to [0, 1]$ be the natural second coordinate projection. Let $X$ be a fan such that $E(X)$ is dense in $X$. By the result of Carruth [1], we may assume that $X$ is embedded in $T$. There is a natural monotone mapping $\pi: X \setminus \{v\} \to C$ such that $e \in [v, \pi(e)]$ for every $e \in E(X)$. We may assume that $\pi(E(X))$ is dense in $C$. The assertion of the following lemma is a consequence of the density of $E(X)$. 
Lemma 3. For every point \( e \in E(X) \) there is a sequence \( \{e_n \in E(X) : n = 1, 2, \ldots \} \) such that \( \pi(e) = \lim_{n \to \infty} \pi(e_n) \) and \( \mathrm{cl}\{e_n : n = 1, 2, \ldots \} = [v, e] \cup \{e_n : n = 1, 2, \ldots \} \).

For a subset \( A \) of \( C \) define
\[
h(A) = \sup\{\rho(e) : e \in E(X) \text{ and } \pi(e) \in A\}.
\]
Without loss of generality we may assume that \( h(C) = 1 \). Let us omit the easy proof of the following lemma.

Lemma 4. For every nonempty closed-and-open subset \( U \) of \( C \) there is a point \( e \in E(X) \) such that \( \pi(e) \in U \) and \( \rho(e) = h(U) \).

Lemma 5. For every \( \varepsilon > 0 \), nonempty closed-and-open subset \( U \) of \( C \) and point \( e \in E(X) \) such that \( \pi(e) \in U \) and \( \rho(e) = h(U) \), there is a null partition \( \{U_n : n = 1, 2, \ldots \} \) of \( U \setminus \{\pi(e)\} \) into closed-and-open subsets of \( C \) of diameter less than \( \varepsilon \) such that \( \{h(U_n) : n = 1, 2, \ldots \} \) is dense in \( \rho([v, e]) \). Moreover, if \( e_n \in E(X) \) is such that \( \pi(e_n) \in U_n \) and \( \rho(e_n) = h(U_n) \), then \( \mathrm{cl}\{e_n : n = 1, 2, \ldots \} = [v, e] \cup \{e_n : n = 1, 2, \ldots \} \).

Proof. By Lemma 3, there is a sequence \( f_m \in E(X) \) such that \( \pi(e) = \lim_{m \to \infty} \pi(f_m) \) and \( \mathrm{cl}\{f_m : m = 1, 2, \ldots \} = [v, e] \cup \{f_m : m = 1, 2, \ldots \} \). Let \( \mathcal{T} = \{W_m : m = 1, 2, \ldots \} \) be a sequence of disjoint closed-and-open subsets of \( U \setminus \{\pi(e)\} \) such that \( \pi(f_m) \in W_m \), \( \mathrm{diam} W_m < \varepsilon \), \( \lim_{m \to \infty} \mathrm{diam} W_m = 0 \), and \( \|\rho(f_m) - h(W_m)\| < 1/m \). Observe that \( \{h(W_m) : m = 1, 2, \ldots \} \) is dense in \( \rho([v, e]) \). Hence, any completion of \( \mathcal{T} \) to a null partition of \( U \setminus \{\pi(e)\} \) with mesh less than \( \varepsilon \) will satisfy the assertion of the lemma.

Construction of an inverse sequence \( \langle T_n, p^n \rangle \) associated with \( X \)

Choose \( e_0 \in E(X) \) such that \( \rho(e_0) = h(C) \). Let \( T_0 = [v, e_0] \) and let \( p_0 : X \to T_0 \) be the horizontal projection onto \( T_0 \). Since \( \rho(e_0) = h(C) \), the map \( p_0 \) is well defined. We will call \( e_0 \) the endpoint of \( T_0 \) and will write \( E(T_0) = \{e_0\} \). Let \( \mathcal{S}_0 = \{C\} \) and let \( \mathcal{S}_1 = \{U_n : n = 1, 2, \ldots \} \) be a partition of \( C \setminus \{\pi(e_0)\} \) guaranteed by Lemma 5 for \( e = e_0 \) and \( \varepsilon = 1 \). For every \( n \) choose \( e_n \in E(X) \) such that \( \pi(e_n) \in U_n \) and \( \rho(e_n) = h(U_n) \). Let \( T_1 = T_0 \cup \{[v, e_n] : n = 1, 2, \ldots \} \). We may define \( p_1 : X \to T_1 \) in such a way that \( p_1|T_0 \) is the identity and \( p_1|\pi^{-1}(U_n) \) is the horizontal projection into \([v, e_n]\). Suppose we have already defined sets \( T_0, \ldots, T_n \subset X \), mappings \( p_k : X \to T_k \) for \( k = 0, \ldots, n \), and collections \( \mathcal{S}_0, \ldots, \mathcal{S}_n \) such that

1. \( T_k \) is a fan for \( k = 1, 2, \ldots, n \),
2. \( T_0 \subset T_1 \subset \cdots \subset T_n \),
3. \( E(T_0) \subset E(T_1) \subset \cdots \subset E(T_n) \),
4. \( p_{k+1}|T_k \) is the identity for \( k = 0, \ldots, n - 1 \),
5. \( \mathcal{S}_{k+1} \) is a null family of disjoint closed-and-open sets of diameter less than or equal to \( 1/(k + 1) \), refining \( \mathcal{S}_k \) and such that \( \bigcup \mathcal{S}_{k+1} = C \setminus \pi(E(T_k)) \),
(6) for every $U \in \mathcal{F}_k$ there is a unique point $e_k(U) \in E(T_k)$ such that 
\[ \pi(e_k(U)) \in U; \] further,
(a) $\rho(e_k(U)) = h(U)$,
(b) $p_k[\pi^{-1}(U)]$ is the horizontal projection into $[v,e_k(U)]$ and 
(c) if $W \in \mathcal{F}_{k-1}$ and $\mathcal{F} = \{ U \in \mathcal{F}_k : U \subset W \}$, then 
\[ \overline{\{e_k(U) : U \in \mathcal{F} \}} = \{e_k(U) : U \in \mathcal{F} \} \cup [v,e_{k-1}(W)]. \]

Observe that in view of Lemmas 4 and 5 the induction can be continued. Define $p^n_m = p_m|T_n$ for $m \leq n$ and observe that $p_m = p^n_m \circ p_n$ and $\rho(x) = \rho \circ p_n(x)$ for every $n$ and $x \in X$. Since $X = \overline{\bigcup \{T_n : n = 0, 1, \ldots \}}$ and every $p_n : X \to T_n$ is a $1/n$-map, the space $X$ is homeomorphic to $\lim \langle T_n, p^n_m \rangle$ and the maps $p_n$ converge to the identity on $X$. We will say that the above inverse sequence is associated with the fan $X$.

**Construction of a homeomorphism**

Now, let $X$ and $Y$ be smooth fans having a dense set of endpoints and let $\langle P_n, p^n_m \rangle$ and $\langle Q_n, q^n_m \rangle$ be inverse sequences associated with $X$ and $Y$, respectively. To complete the proof of the theorem we will construct a sequence of homeomorphisms $h_n : P_n \overset{\text{onto}}{\to} Q_n$ inducing a homeomorphism between the limit spaces.

Let $v$ be the vertex of $X$ and $w$ the vertex of $Y$. We may assume that $\rho(X) = \rho(Y) = [0,1]$. Let $h_0 : P_0 \overset{\text{onto}}{\to} Q_0$ be the linear map such that $h_0(v) = w$. Let $\mathcal{F}_0 = \{h_0\}$. Let $\{e_n : n = 1, 2, \ldots \} = E(P_1) \setminus E(P_0)$ and $\{f_n : n = 1, 2, \ldots \} = E(Q_1) \setminus E(Q_0)$. We may find a permutation $\varphi$ of positive integers such that 
\[ |\rho(e_n) - \rho(f_{\varphi(n)})| \leq \min \left\{ \frac{1}{\alpha(n)\sqrt{5}}, \frac{\rho(e_n)}{4} \right\} \]
for every $n$, where $\alpha(n) = \min\{n, \varphi(n)\}$.

Let $h_1 : P_1 \overset{\text{onto}}{\to} Q_1$ be the extension of $h_0$, such that $h_1$ maps $[v,e_1]$ linearly onto $[w,f_{\varphi(1)}]$ for every $n$. Observe that if $x \in P_1 \setminus \{v\}$, then $\pi \circ q_0 \circ h_1(x) = \pi \circ h_0 \circ p_1(x)$. Hence, $|q_0 \circ h_1(z) - h_0 \circ p_1(z)| \leq 1/2$ for every $z$.

For each nonnegative integer $n$ we will inductively construct a class $\mathcal{F}_n$ of homeomorphisms mapping $P_n$ onto $Q_n$ such that for every $h \in \mathcal{F}_n$ and $\varepsilon > 0$ there is an $g \in \mathcal{F}_{n+1}$ satisfying the following conditions:

(7) $g|P_n = h$,

(8) for every $e \in E(P_n)$ the function $h|[v,e] : [v,e] \overset{\text{onto}}{\to} [w,h(e)]$ is a linear homeomorphism,

(9) $\pi \circ q_{n+1} \circ g(x) = \pi \circ h \circ p_{n+1}(x)$,

(10) $|\rho \circ q_{n+1} \circ g(x) - \rho \circ h \circ p_{n+1}(x)| < \varepsilon$, and

(11) $|\rho(x) - \rho \circ h(x)| < \rho(x)/4$ for each $x \in P_n$.

Suppose that classes of homeomorphisms $\mathcal{F}_i$ satisfying (7)–(11) have already been defined for each $i$, $0 \leq i \leq n$. Let $\mathcal{U}_i[\mathcal{V}_i]$ be the partition
of $C \setminus E(P_{i-1})$ [of $C \setminus E(Q_{i-1})$, respectively] used for the construction of $P_i(Q_i)$, where $i = 1, 2, \ldots, n$. Let $\{U_k : k = 1, 2, \ldots\}$ be an enumeration of $\mathcal{U}_n$ and let $e_k = e(U_k)$. The homeomorphism $h : P_n \to Q_n$ maps each $e_k$ to a point $f_k \in E(Q_n)$. Let $V_k$ be the unique element in $\mathcal{V}_n$ containing $f_k$. Then $\{V_k : k = 1, 2, \ldots\}$ is an enumeration of $\mathcal{V}_n$. For every $k$, let $\mathcal{V}_{n+1}(U_k) = \{U \in \mathcal{V}_{n+1} : U \subset U_k\}$ and $\mathcal{V}_{n+1}(V_k) = \{V \in \mathcal{V}_{n+1} : V \subset V_k\}$. Let $\{U_{k,j} : j = 1, 2, \ldots\}$ and $\{V_{k,j} : j = 1, 2, \ldots\}$ be enumerations of $\mathcal{V}_{n+1}(U_k)$ and $\mathcal{V}_{n+1}(V_k)$, respectively. Put $e_{k,j} = e(U_{k,j})$ and $f_{k,j} = f(V_{k,j})$. Recall that 

$$
\rho(x) = \rho \circ p_n(x).
$$

By (11), $|\rho(e_{k,j}) - \rho \circ h \circ p_n^{n+1}(e_{k,j})| < \frac{\rho(e_{k,j})}{4}.

For every $k$ we may find a permutation $\phi_k$ of positive integers such that

$$
(\ast) \quad |\rho(e_{k,j}) - \rho(f_{k,j})| < \frac{\rho(e_{k,j})}{4}
$$

and

$$
(\ast \ast) \quad |\rho \circ h \circ p_n^{n+1}(e_{k,j}) - \rho(f_{k,j})| < \min \left\{ e, \frac{1}{\alpha(j) + k} \right\},
$$

where $\alpha(j) = \min\{j, \phi_k(j)\}$. Let $g$ be the extension of $h$ which maps $[v, e_{k,j}]$ linearly onto $[w, f_{k,j}]$. Then (7) and (8) follow immediately; (10) and the continuity of $g$ follow from ($\ast \ast$) and the linearity of $g$; and (11) for $g$ follows from (11) for $h$, the condition ($\ast$), and the fact that $h[v, e_{k,j}]$ is linear.

Let $\mathcal{F}_{n+1}$ be the class of homeomorphisms $g$ mapping $P_{n+1}$ onto $Q_{n+1}$ obtained as described above for every $h \in \mathcal{F}_n$ and $e = 1/r$, where $r = 1, 2, \ldots$.

By Lemma 1, we can select a sequence $h_n \in \mathcal{F}_n$ of homeomorphisms such that for each $k \leq n \leq m$ diagram ($\ast$) is $1/2^n$-commutative. Hence the homeomorphisms $h_n$ induce a continuous map $h : X \to Y$ defined by $h(x) = y$, where $q_\infty(y) = \lim_{n \to \infty} g_n \circ h_n \circ p_n(x)$. To complete the proof it suffices to show that $h$ is one-to-one.

Let $x \neq a \in X$ and let $x_n = p_n(x)$, $a_n = p_n(a)$, $h(x) = y$, and $h(a) = b$. Suppose first that for some $n$, $x \neq p_n(a)$. By (9), the definition of $h$ and the fact that each $h_n$ is a homeomorphism, $\pi \circ q_n \circ h(x) \neq \pi \circ q_n \circ h(a)$, and $h(x) \neq h(a)$. Hence we may assume that either $\pi(x_n) = \pi(a_n)$, for each $n$ or $a = v$.

Then there exists a (unique) $e \in E(P_n)$ such that $p_n([v, e]) \subset [v, e_n]$. Since $e \in E(X)$, $\lim_{n \to \infty} [v, e_n] = [v, e]$ and $\lim_{n \to \infty} \rho(e_n) = \rho(e)$. Clearly, $\rho(e_{n+1}) \leq \rho(e_n)$ for each $n$. By (11),

$$
\rho \circ h_n(e_n) > \frac{3}{4} \rho(e) 
$$

for each $n$. Since each $h_n$ is linear,

$$
|\rho \circ h_n(x_n) - \rho \circ h_n(y_n)| \geq \frac{3}{4} |\rho(x_n) - \rho(y_n)|
$$

for each $n$. Therefore
for each \( n \) and \( x_n, y_n \in [v, e_n] \). Since \( \lim_{n \to \infty} [v, e_n] = [v, e] \), there exist
\( c_n, d_n \in [v, e_n] \) such that \( \lim_{n \to \infty} c_n = x \) and \( \lim_{n \to \infty} d_n = a \). Hence,
\[
\lim_{n \to \infty} |\rho \circ h_n(c_n) - \rho \circ h_n(d_n)| \geq \lim_{n \to \infty} |\rho(c_n) - \rho(d_n)| = |\rho(x) - \rho(a)| > 0.
\]
Hence, by Lemma 2, \( |\rho \circ h(x) - \rho \circ h(a)| > 0 \) and \( h(x) \neq h(a) \).

Added in proof. The main result of this paper was proved independently by W. J. Charatonik [The Lelek fan is unique, (to appear in Houston J. of Math.)].

REFERENCES

4. J. C. Mayer, An explosion point for the set of endpoints of the Julia set of \( \lambda \exp(z) \), (to appear in Ergodic Theory and Dynamical Systems).