

A FUNCTION SPACE TRIPLE OF A COMPACT POLYHEDRON INTO AN OPEN SET IN EUCLIDEAN SPACE

KATSURO SAKAI

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ABSTRACT. Let X be a non-zero dimensional compact Euclidean polyhedron and Y an open set in Euclidean space \mathbf{R}^r ($r > 0$). The spaces of (continuous) maps, Lipschitz maps and PL maps from X to Y are denoted by $C(X, Y)$, $LIP(X, Y)$ and $PL(X, Y)$, respectively. We prove that the triple

$$(C(X, Y), LIP(X, Y), PL(X, Y))$$

is an (s, Σ, σ) -manifold triple, where $s = (-1, 1)^\omega$,

$$\Sigma = \{x \in s \mid \sup |x(i)| < 1\} \text{ and}$$

$$\sigma = \{x \in s \mid x(i) = 0 \text{ except for finitely many } i\}.$$

0. INTRODUCTION

A paracompact (topological) manifold modelled on a given space E is called an E -manifold. For $F \subset E$, an (E, F) -manifold pair is a pair (M, N) of an E -manifold M and an F -manifold N which admits an open cover \mathcal{U} of M and open embeddings $\varphi_U: U \rightarrow E$, $U \in \mathcal{U}$, such that $\varphi_U(U \cap N) = \varphi_U(U) \cap F$. For $G \subset F \subset E$, an (E, F, G) -manifold triple is defined by the same way. Let Q denote the Hilbert cube $[-1, 1]^\omega$, $s = (-1, 1)^\omega$ the pseudo-interior of Q ,

$$\Sigma = \{x \in s \mid \sup |x(i)| < 1\} \text{ and}$$

$$\sigma = \{x \in s \mid x(i) = 0 \text{ except for finitely many } i\},$$

where $x(i)$ denotes the i -th coordinate of x . Note that s is homeomorphic to (\cong) the Hilbert space ℓ_2 [An]. Moreover it is proved in [SW₂] that $(s, \Sigma, \sigma) \cong (\ell_2, \ell_2^Q, \ell_2^f)$, where ℓ_2^Q is the linear span of the Hilbert cube $\prod_{i \in \mathbf{N}} [-i^{-1}, i^{-1}]$ in ℓ_2 and ℓ_2^f is the linear span of the usual orthonormal basis of ℓ_2 , that is,

$$\ell_2^Q = \{x \in \ell_2 \mid \sup |i \cdot x(i)| < \infty\} \text{ and}$$

$$\ell_2^f = \{x \in \ell_2 \mid x(i) = 0 \text{ except for finitely many } i\}.$$

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Let $X = (X, d)$ be a non-discrete metric compactum and $Y = (Y, \rho)$ be a separable metric space without isolated points. The space of (continuous) maps from X to Y is denoted by $C(X, Y)$. The topology of $C(X, Y)$ is induced by the sup-metric

$$\rho(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

Then it is known that $C(X, Y)$ is an s -manifold (i.e., l_2 -manifold) if Y is a complete ANR [Ge₁, To, Sa₁].

By $\text{LIP}(X, Y)$, we denote the subspace of $C(X, Y)$ consisting of all Lipschitz maps. The Lipschitz constant of $f \in \text{LIP}(X, Y)$ is denoted by $\text{lip } f$, i.e.,

$$\text{lip } f = \inf\{k \geq 0 \mid \rho(f(x), f(y)) \leq k \cdot d(x, y)\}.$$

For each $k > 0$, let

$$\begin{aligned} k\text{-LIP}(X, Y) &= \{f \in \text{LIP}(X, Y) \mid \text{lip } f \leq k\} \text{ and} \\ \text{LIP}_k(X, Y) &= \{f \in \text{LIP}(X, Y) \mid \text{lip } f < k\} = \bigcup_{k' < k} k'\text{-LIP}(X, Y). \end{aligned}$$

Then $(C(X, Y), \text{LIP}(X, Y))$ is an (s, Σ) -manifold pair if Y is a locally compact, locally convex set in a normed linear space [SW₁] or a Lipschitz manifold or a Euclidean polyhedron [Sa₂]. For each $k > 0$,

$$(k\text{-LIP}(X, Y), \text{LIP}_k(X, Y)) \cong (Q, \Sigma)$$

if Y is a compact convex set in a normed linear space [SW₁].

In case X and Y are polyhedra, Geoghegan [Ge₂] proved that $(C(X, Y), \text{PL}(X, Y))$ is an (s, σ) -manifold pair, where $\text{PL}(X, Y)$ denotes the subspace of $C(X, Y)$ consisting of all PL maps. It is natural to conjecture that the triple

$$(C(X, Y), \text{LIP}(X, Y), \text{PL}(X, Y))$$

is an (s, Σ, σ) -manifold triple. In this paper, we prove this conjecture in the case where Y is an open set in Euclidean space \mathbf{R}^r (cf. [SW₁, Conjecture 3.3]).

Main result. *Let X be a non-zero dimensional compact polyhedron and Y an open set in Euclidean space \mathbf{R}^r ($r > 0$). Then the triple*

$$(C(X, Y), \text{LIP}(X, Y), \text{PL}(X, Y))$$

is an (s, Σ, σ) -manifold triple.

This is obtained as a corollary of the following:

0.1. **Theorem.** *Let X be a non-zero dimensional compact polyhedron and Y a non-degenerate locally compact convex set in Euclidean space \mathbf{R}^r . Then*

$$(C(X, Y), \text{LIP}(X, Y), \text{PL}(X, Y)) \cong (s, \Sigma, \sigma).$$

1. PRELIMINARIES

A closed set A in a metric space $W = (W, d)$ is a Z -set if for each map $f: Q \rightarrow W$ and each $\varepsilon > 0$, there is a map $g: Q \rightarrow W \setminus A$ with $d(f, g) < \varepsilon$. A countable union of Z -sets in W is called a Z_σ -set. A subset M of W is a *cap* (or *fd-cap*) set for W if $M = \bigcup_{n \in \mathbb{N}} M_n$, where $M_1 \subset M_2 \subset \dots$ is a tower of (resp. finite-dimensional) compact Z -sets in W with the property

(fd)-cap: For each (finite-dimensional) compact set A in W , each $\varepsilon > 0$ and each $m \in \mathbb{N}$, there is an $n \geq m$ ($\in \mathbb{N}$) and an embedding $h: A \rightarrow M_n$ such that $h|_A \cap M_m = \text{id}$ and $d(h, \text{id}) < \varepsilon$.

or (fd)-cap sets, refer to [Ch].

The following characterization of (s, Σ, σ) -(or (Q, Σ, σ))-manifold triples is given in [SW₂].

1.1. Theorem. *A triple (W, M, N) of spaces is an (s, Σ, σ) -(or (Q, Σ, σ))-manifold triple if and only if W is an s -(or Q)-manifold, and there is a tower $M_1 \subset M_2 \subset \dots$ of compact Q -manifolds in W with **cap** such that $M = \bigcup_{i \in \mathbb{N}} M_i$, each M_i is a Z -set in M_{i+1} and each $M_i \cap N$ is an fd-cap set for M_i .*

A map $f: X \rightarrow Y$ between metric spaces is *locally Lipschitz* (abbreviated *LIP*) if each $x \in X$ has a neighborhood U such that $f|_U$ is Lipschitz. Notice, each LIP map $f: X \rightarrow Y$ is Lipschitz if X is compact. A metric space Y is an *absolute LIP extensor* (abbreviated *ALE*) if every LIP map $f: A \rightarrow Y$ of a closed set of an arbitrary metric space X has a LIP extension $\hat{f}: X \rightarrow Y$.

1.2. Lemma. *Any compact convex set Y in Euclidean space \mathbf{R}^r is an ALE.*

Proof. By [Wo, Lemma 4], the nearest point retraction $\nu: \mathbf{R}^r \rightarrow Y$ defined by

$$\|x - \nu(x)\| = \text{dist}(x, Y) = \inf\{\|x - y\| \mid y \in Y\}$$

is Lipschitz, where $\|\cdot\|$ is the Euclidean norm. Since \mathbf{R}^r is an ALE by [LV, Theorem 5.7], Y is also an ALE. \square

2. LIPSCHITZ CONSTANTS OF PL MAPS

For each $x \in \mathbf{R}^n$, $x(i)$ denotes the i -th coordinate of x . The Euclidean norm of \mathbf{R}^n is Lipschitz equivalent to the norm defined as follows:

$$\|x\| = |x(1)| + \dots + |x(n)| \quad \text{for } x \in \mathbf{R}^n.$$

Let $\{e_1, \dots, e_q\}$ be the canonical orthonormal basis for \mathbf{R}^q , that is, $e_i(j) = 0$ if $i \neq j$ and $e_i(i) = 1$. The q -th symmetric group is denoted by Σ_q . For each $\alpha = (\alpha(1), \dots, \alpha(q)) \in \Sigma_q$, $\Delta(\alpha)$ denotes the full simplicial complex with vertices

$$\begin{aligned} v_0^\alpha &= 0, & v_1^\alpha &= e_{\alpha(1)}, \\ v_2^\alpha &= e_{\alpha(1)} + e_{\alpha(2)}, \dots, & v_q^\alpha &= e_{\alpha(1)} + \dots + e_{\alpha(q)}. \end{aligned}$$

2.1. **Lemma.** For each linear (affine) map $f: |\Delta(\alpha)| \rightarrow \mathbf{R}$,

$$\text{lip } f = \max\{|f(v_i^\alpha) - f(v_{i-1}^\alpha)| \mid i = 1, \dots, q\}.$$

Proof. First observe that for each $x \in |\Delta(\alpha)|$,

$$1 \geq x(\alpha(1)) \geq x(\alpha(2)) \geq \dots \geq x(\alpha(q)) \geq 0$$

and

$$\begin{aligned} x &= (1 - x(\alpha(1))) \cdot v_0^\alpha + (x(\alpha(1)) - x(\alpha(2))) \cdot v_1^\alpha + \dots \\ &\quad + (x(\alpha(q-1)) - x(\alpha(q))) \cdot v_{q-1}^\alpha + x(\alpha(q)) \cdot v_q^\alpha. \end{aligned}$$

Since f is linear, it follows that

$$\begin{aligned} f(x) &= (1 - x(\alpha(1))) \cdot f(v_0^\alpha) + (x(\alpha(1)) - x(\alpha(2))) \cdot f(v_1^\alpha) + \dots \\ &\quad + (x(\alpha(q-1)) - x(\alpha(q))) \cdot f(v_{q-1}^\alpha) + x(\alpha(q)) \cdot f(v_q^\alpha). \end{aligned}$$

Then for each $x, x' \in |\Delta(\alpha)|$,

$$\begin{aligned} |f(x) - f(x')| &\leq |x(\alpha(1)) - x'(\alpha(1))| \cdot |f(v_1^\alpha) - f(v_0^\alpha)| + \dots \\ &\quad + |x(\alpha(q)) - x'(\alpha(q))| \cdot |f(v_q^\alpha) - f(v_{q-1}^\alpha)| \\ &\leq \|x - x'\| \cdot \max\{|f(v_i^\alpha) - f(v_{i-1}^\alpha)| \mid i = 1, \dots, q\}. \end{aligned}$$

Thus we have the result. \square

For each $c > 0$ and $v \in \mathbf{R}^q$, let $h_{c,v}: \mathbf{R}^q \rightarrow \mathbf{R}^q$ be the (affine) homeomorphism defined by $h_{c,v}(x) = c \cdot x + v$. For each $p \in \mathbf{N}$, we define the triangulation $K(q, p)$ of $I^q = [0, 1]^q$ as follows:

$$K(q, p) = \{h_{1/p,v}(\sigma) \mid \sigma \in \Delta(\alpha), \alpha \in \Sigma_q, v \in V_p^q\},$$

where

$$V_p^q = \left\{0, \frac{1}{p}, \dots, \frac{p-1}{p}\right\}^q \subset I^q.$$

2.2. **Lemma.** Let $f: I^q \rightarrow \mathbf{R}$ be a map such that $f|_\sigma$ is linear for each $\sigma \in K(q, p)$. Then

$$\text{lip } f = \text{lip}(f|_{K(q, p)^0}).$$

Proof. Let $k = \text{lip}(f|_{K(q, p)^0})$. It suffices to show that $\text{lip } f \leq k$. First we show that $\text{lip}(f|_\sigma) \leq k$ for each $\sigma \in K(q, p)$. Choose an $\alpha \in \Sigma_q$ and $v \in V_p^q$ so that $h_{1/p,v}^{-1}(\sigma) \in \Delta(\alpha)$. Observe that $\text{lip}(h_{1/p,v}^{-1}) = p$ and

$$\|h_{1/p,v}(v_i^\alpha) - h_{1/p,v}(v_{i-1}^\alpha)\| = \frac{1}{p}$$

for each $i = 1, \dots, q$. By Lemma 2.1, we have

$$\begin{aligned} \text{lip}(f|_\sigma) &\leq \text{lip}(f \circ h_{1/p,v}^{-1} |_{|\Delta(\alpha)|}) \cdot \text{lip}(h_{1/p,v}^{-1} |_\sigma) \\ &\leq p \cdot \max\{|f h_{1/p,v}(v_i^\alpha) - f h_{1/p,v}(v_{i-1}^\alpha)| \mid i = 1, \dots, q\} \\ &\leq p \cdot \max\{k \cdot \|h_{1/p,v}(v_i^\alpha) - h_{1/p,v}(v_{i-1}^\alpha)\| \mid i = 1, \dots, q\} \\ &= k. \end{aligned}$$

Now we show $\text{lip } f \leq k$. Let $x \neq x' \in I^q = |K(q, p)|$. If x and x' are contained in same $\sigma \in K(q, p)$, then

$$|f(x) - f(x')| \leq \text{lip}(f|\sigma) \cdot \|x - x'\| \leq k \cdot \|x - x'\|.$$

Otherwise choose points x_0, x_1, \dots, x_n on the straight line segment connecting x and x' so that $x_0 = x$, $x_n = x'$, and each pair x_{i-1} and x_i are contained in some $\sigma_i \in K(q, p)$. Since

$$\|x - x'\| = \|x_0 - x_1\| + \|x_1 - x_2\| + \dots + \|x_{n-1} - x_n\|,$$

we have

$$\begin{aligned} |f(x) - f(x')| &\leq |f(x_0) - f(x_1)| + \dots + |f(x_{n-1}) - f(x_n)| \\ &\leq \text{lip}(f|\sigma_1) \cdot \|x_0 - x_1\| + \dots + \text{lip}(f|\sigma_n) \cdot \|x_{n-1} - x_n\| \\ &\leq k \cdot \|x - x'\|. \end{aligned}$$

This completes the proof. \square

2.3. Corollary. Let $f: I^q \rightarrow \mathbf{R}^r$ be a map such that $f|\sigma$ is linear for each $\sigma \in K(q, p)$. Then

$$\text{lip } f \leq r \cdot \text{lip}(f|K(q, p))^0.$$

Proof. For each $i = 1, \dots, r$, let $\pi_i: \mathbf{R}^r \rightarrow \mathbf{R}$ denote the projection onto the i -th coordinates. By Lemma 2.2,

$$\begin{aligned} \text{lip } \pi_i f &= \text{lip}(\pi_i f|K(q, p))^0 \\ &\leq (\text{lip } \pi_i) \cdot (\text{lip}(f|K(q, p))^0) \\ &\leq \text{lip}(f|K(q, p))^0. \end{aligned}$$

Then for each $x, x' \in I^q$,

$$\begin{aligned} \|f(x) - f(x')\| &= |\pi_1 f(x) - \pi_1 f(x')| + \dots + |\pi_r f(x) - \pi_r f(x')| \\ &\leq (\text{lip } \pi_1 f + \dots + \text{lip } \pi_r f) \cdot \|x - x'\| \\ &\leq r \cdot \text{lip}(f|K(q, p))^0 \cdot \|x - x'\|. \quad \square \end{aligned}$$

2.4. Lemma. Let Y be a convex set in Euclidean space \mathbf{R}^r . Then for each $q \in \mathbf{N}$, $k > 0$ and $\varepsilon > 0$, there is a map

$$\varphi: k\text{-LIP}(I^q, Y) \rightarrow \text{PL}(I^q, Y) \cap rk\text{-LIP}(I^q, Y)$$

which is ε -close to id .

Proof. Choose $p \in \mathbf{N}$ so large that

$$\text{mesh } K(q, p) < \varepsilon / (r + 1)k.$$

Since Y is convex, the desired map φ can be defined as follows:

$$\varphi(f)|K(q, p)^0 = f|K(q, p)^0$$

and

$$\varphi(f)|\sigma \text{ is linear for each } \sigma \in K(q, p).$$

In fact, $\text{lip } \varphi(f) \leq r \cdot \text{lip } f \leq rk$ by the above corollary. For each $x \in I^q$, we have a $v \in K(q, p)^0$ such that $\|x - v\| < \varepsilon/(r+1)k$. Then

$$\begin{aligned} \|\varphi(f)(x) - f(x)\| &\leq \|\varphi(f)(x) - \varphi(f)(v)\| + \|f(x) - f(v)\| \\ &\leq rk \cdot \|x - v\| + k \cdot \|x - v\| \\ &\leq \varepsilon. \end{aligned}$$

Thus φ is ε -close to id . \square

3. SPACE OF k -LIPSCHITZ PL MAPS

Let X be a non-zero dimensional compact polyhedron in Euclidean space \mathbf{R}^q and Y a non-degenerate locally compact convex set in Euclidean space \mathbf{R}^r . Then we can use the metrics for X and Y defined by the norm in §2 instead of Euclidean norm to prove the theorem. This norm is of great advantage to our problem as seen in Section 2 and since the map $\|\cdot\|: \mathbf{R}^q \rightarrow \mathbf{R}$ is PL. Let us regard $C(X, Y)$ as a subset of the Banach space $C(X, \mathbf{R}^r)$ with the sup-norm

$$\|f\| = \sup\{\|f(x)\| \mid x \in X\}.$$

Then note that $\text{PL}(X, Y) \subset C(X, Y)$ and each $k\text{-LIP}(X, Y)$ are convex sets in $C(X, \mathbf{R}^r)$.

3.1. Lemma. *For each $k > 0$, $\text{PL}(X, Y) \cap k\text{-LIP}(X, Y)$ is infinite dimensional.*

Proof. Without loss of generality, we may assume $0 \in Y$. Let K be a triangulation of X and $v_0 \in K^0$ be a non-isolated vertex. For each $n \in \mathbf{N}$, choose any $v_n \in \text{Lk}(v_0, \text{Sd}^n K)^0$, where $\text{Sd}^n K$ is the n -th barycentric subdivision of K . Let $g_n \in \text{PL}(X, Y)$ so that $g_n(v) = 0$ for $v \in (\text{Sd}^n K)^0 \setminus \{v_n\}$, $g_n(v_n) \neq 0$ and $g_n|_\sigma$ is linear for each $\sigma \in \text{Sd}^n K$. By choosing $g_n(v_n)$ close to 0, we have

$$g_n \in \text{PL}(X, Y) \cap k\text{-LIP}(X, Y).$$

It is easy to verify that $\{g_n\}_{n \in \mathbf{N}}$ are linearly independent. Thus $\text{PL}(X, Y) \cap k\text{-LIP}(X, Y)$ is infinite-dimensional. \square

By [Ge₂, Lemma 4.3], $\text{PL}(X, Y)$ is σ -fd-compact, that is, a countable union of finite-dimensional compacta. Then $\text{PL}(X, Y) \cap k\text{-LIP}(X, Y)$ is an infinite-dimensional σ -fd-compact convex set in the Banach space $C(X, \mathbf{R}^r)$. If Y is compact,

$$\begin{aligned} &\text{cl}_{C(X, \mathbf{R}^r)}(\text{PL}(X, Y) \cap k\text{-LIP}(X, Y)) \\ &= \text{cl}_{C(X, Y)}(\text{PL}(X, Y) \cap k\text{-LIP}(X, Y)) \subset k\text{-LIP}(X, Y), \end{aligned}$$

which are compact by Arzela-Ascoli's Theorem. By [Do, Theorem 2], we have the following theorem.

3.2. **Theorem.** *If Y is compact, then, for each $k > 0$,*

$$(\text{cl}_{C(X,Y)}(\text{PL}(X, Y) \cap k\text{-LIP}(X, Y)), \text{PL}(X, Y) \cap k\text{-LIP}(X, Y)) \cong (Q, \sigma). \quad \square$$

In the above, it is natural to ask whether or not

$$\text{cl}_{C(X,Y)}(\text{PL}(X, Y) \cap k\text{-LIP}(X, Y)) = k\text{-LIP}(X, Y).$$

The author has not succeeded in giving its proof nor a counterexample. This is related to Problem 4.1.

The following lemma follows from the proof of [SW₁, Lemma 1.2]:

3.3. **Lemma.** *For each $k > 0$ and $\varepsilon > 0$, there exists a map*

$$\psi: k\text{-LIP}(X, Y) \rightarrow k\text{-LIP}(X, Y) \setminus \text{LIP}_k(X, Y)$$

such that ψ is ε -close to id and

$$\psi(\text{PL}(X, Y) \cap k\text{-LIP}(X, Y)) \subset \text{PL}(X, Y) \cap k\text{-LIP}(X, Y).$$

Proof. In fact, we use the norm $\|\cdot\|$ in §2 which is PL. Then it is easily observed that the map ψ defined in [SW₁, Lemma 1.2] maps $\text{PL}(X, Y) \cap k\text{-LIP}(X, Y)$ into itself. \square

3.4. **Lemma.** *For each compact set $A \subset C(X, Y)$ and $\varepsilon > 0$, there is a $k > 0$ and a map*

$$\varphi: A \rightarrow \text{PL}(X, Y) \cap k\text{-LIP}(X, Y)$$

which is ε -close to id.

Proof. Without loss of generality, we can assume that $X \subset I^q$. Let $R: C(I^q, Y) \rightarrow C(X, Y)$ be the restriction map, that is, $R(f) = f|_X$ for each $f \in C(I^q, Y)$. Note that for any $k > 0$,

$$R(\text{PL}(I^q, Y) \cap k\text{-LIP}(I^q, Y)) \subset \text{PL}(X, Y) \cap k\text{-LIP}(X, Y).$$

Since Y is an AR, there is an extension map $E: C(X, Y) \rightarrow C(I^q, Y)$, namely a right inverse of R , i.e., $R \circ E = \text{id}$. In fact, the evaluation $e: X \times C(X, Y) \rightarrow Y$ extends to a map $\tilde{e}: I^q \times C(X, Y) \rightarrow Y$. Then E can be defined by $E(f)(x) = \tilde{e}(x, f)$. By [SW₁, Lemma 1.4], we have a $k' > 0$ and a map $\varphi': E(A) \rightarrow k'\text{-LIP}(I^q, Y)$ which is $\varepsilon/2$ -close to id. Let $k = rk' > 0$. By Lemma 2.4, we have a map

$$\psi: k'\text{-LIP}(I^q, Y) \rightarrow \text{PL}(I^q, Y) \cap k\text{-LIP}(I^q, Y)$$

which is $\varepsilon/2$ -close to id. Then the composition

$$\varphi = R \circ \psi \circ \varphi' \circ E: A \rightarrow \text{PL}(X, Y) \cap k\text{-LIP}(X, Y)$$

is the desired map. In fact, for each $f \in A$ and $x \in X$,

$$\begin{aligned} \|\varphi(f)(x) - f(x)\| &= \|\psi(\varphi'(E(f)))(x) - E(f)(x)\| \\ &\leq \|\psi(\varphi'(E(f)))(x) - \varphi'(E(f))(x)\| \\ &\quad + \|\varphi'(E(f))(x) - E(f)(x)\| \\ &\leq \|\psi - \text{id}\| + \|\varphi' - \text{id}\| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \square \end{aligned}$$

4. PROOF OF THEOREM 0.1

First recall that $C(X, Y) \cong s$. Write $Y = \bigcup_{n \in \mathbb{N}} Y_n$, where each Y_n is a non-degenerate compact convex set and $Y_n \subset \text{int}_Y Y_{n+1}$. For each $n \in \mathbb{N}$, let

$$M_n = \text{cl}_{C(X, Y)}(\text{PL}(X, Y_n) \cap n\text{-LIP}(X, Y_n)).$$

It follows from Lemma 3.3 that each M_n is a Z -set in M_{n+1} . And by Theorem 3.2, $M_n \cong Q$ and

$$\text{PL}(X, Y) \cap M_n = \text{PL}(X, Y) \cap n\text{-LIP}(X, Y_n)$$

is an fd-cap set for M_n .

We will show that $\text{LIP}(X, Y) = \bigcup_{n \in \mathbb{N}} M_n$. Without loss of generality, we can assume that $X \subset I^q$. Let $f \in \text{LIP}(X, Y)$. From compactness, $f(X) \subset Y_{n'}$ for some $n' \in \mathbb{N}$. By Lemma 1.2, f has an extension $\tilde{f} \in \text{LIP}(I^q, Y_{n'})$. Let $n \geq \max\{n', r \cdot \text{lip } \tilde{f}\}$. Then by Lemma 2.4,

$$\tilde{f} \in \text{cl}_{C(I^q, Y)} \text{PL}(I^q, Y_n) \cap n\text{-LIP}(I^q, Y_n).$$

Therefore $f = \tilde{f}|_X \in M_n$.

By using Lemma 3.4, we can prove similarly as [SW₁, Theorem 2.1] that the tower $\{M_n\}_{n \in \mathbb{N}}$ has the property **cap** for $C(X, Y)$. Thus Theorem 0.1 follows from Theorem 1.1. \square

Finally we pose the following problem:

4.1. Problem. If Y is a non-degenerate compact convex set, then for each $k > 0$, is

$$(k\text{-LIP}(X, Y), \text{LIP}_k(X, Y), \text{PL}(X, Y) \cap \text{LIP}_k(X, Y)) \cong (Q, \Sigma, \sigma)?$$

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA-CITY, 305 JAPAN