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A FUNCTION SPACE TRIPLE OF A COMPACT POLYHEDRON INTO AN OPEN SET IN EUCLIDEAN SPACE

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ABSTRACT. Let X be a non-zero dimensional compact Euclidean polyhedron and Y an open set in Euclidean space $\mathbf{R}^r$ $(r > 0)$. The spaces of (continuous) maps, Lipschitz maps and PL maps from X to Y are denoted by $C(X, Y)$, $\mathrm{LIP}(X, Y)$ and $\mathrm{PL}(X, Y)$, respectively. We prove that the triple

$$(C(X, Y), \mathrm{LIP}(X, Y), \mathrm{PL}(X, Y))$$

is an (s, Σ, σ)-manifold triple, where $s = (-1, 1)^\omega$,

$$\Sigma = \{x \in s \mid \sup |x(i)| < 1\} \text{ and}$$
$$\sigma = \{x \in s \mid x(i) = 0 \text{ except for finitely many } i\}.$$

0. INTRODUCTION

A paracompact (topological) manifold modelled on a given space E is called an *E-manifold*. For $F \subset E$, an *(E, F)-manifold pair* is a pair (M, N) of an E-manifold M and an F-manifold N which admits an open cover $\mathscr{U}$ of M and open embeddings $\varphi_U \colon U \to E$, $U \in \mathscr{U}$, such that $\varphi_U(U \cap N) = \varphi_U(U) \cap F$. For $G \subset F \subset E$, an *(E, F, G)-manifold triple* is defined by the same way. Let Q denote the Hilbert cube $[-1, 1]^\omega$, $s = (-1, 1)^\omega$ the pseudo-interior of Q,

$$\Sigma = \{x \in s \mid \sup |x(i)| < 1\} \text{ and}$$
$$\sigma = \{x \in s \mid x(i) = 0 \text{ except for finitely many } i\},$$

where $x(i)$ denotes the i-th coordinate of x. Note that s is homeomorphic to $(\cong)$ the Hilbert space ℓ_2 [An]. Moreover it is proved in [SW$_2$] that $(s, \Sigma, \sigma) \cong (\ell_2, \ell_2^Q, \ell_2^f)$, where ℓ_2^Q is the linear span of the Hilbert cube $\prod_{i \in \mathbf{N}}[-i^{-1}, i^{-1}]$ in ℓ_2 and ℓ_2^f is the linear span of the usual orthonormal basis of ℓ_2, that is,

$$\ell_2^Q = \{x \in \ell_2 \mid \sup |i \cdot x(i)| < \infty\} \text{ and}$$
$$\ell_2^f = \{x \in \ell_2 \mid x(i) = 0 \text{ except for finitely many } i\}.$$

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Let $X = (X, d)$ be a non-discrete metric compactum and $Y = (Y, \rho)$ be a separable metric space without isolated points. The space of (continuous) maps from X to Y is denoted by $C(X, Y)$. The topology of $C(X, Y)$ is induced by the sup-metric

$$\rho(f, g) = \sup\{\rho(f(x), g(x)) | x \in X\}.$$

Then it is known that $C(X, Y)$ is an s-manifold (i.e., l_2-manifold) if Y is a complete ANR [Ge_1, To, Sa_1].

By $\mathrm{LIP}(X, Y)$, we denote the subspace of $C(X, Y)$ consisting of all Lipschitz maps. The Lipschitz constant of $f \in \mathrm{LIP}(X, Y)$ is denoted by $\mathrm{lip}\, f$, i.e.,

$$\mathrm{lip}\, f = \inf\{k \geq 0 | \rho(f(x), f(y)) \leq k \cdot d(x, y)\}.$$

For each $k > 0$, let

$$k\text{-}\mathrm{LIP}(X, Y) = \{f \in \mathrm{LIP}(X, Y) | \mathrm{lip}\, f \leq k\} \text{ and}$$
$$\mathrm{LIP}_k(X, Y) = \{f \in \mathrm{LIP}(X, Y) | \mathrm{lip}\, f < k\} = \bigcup_{k' < k} k'\text{-}\mathrm{LIP}(X, Y).$$

Then $(C(X, Y), \mathrm{LIP}(X, Y))$ is an (s, Σ)-manifold pair if Y is a locally compact, locally convex set in a normed linear space [SW_1] or a Lipschitz manifold or a Euclidean polyhedron [Sa_2]. For each $k > 0$,

$$(k\text{-}\mathrm{LIP}(X, Y), \mathrm{LIP}_k(X, Y)) \cong (Q, \Sigma)$$

if Y is a compact convex set in a normed linear space [SW_1].

In case X and Y are polyhedra, Geoghegan [Ge_2] proved that $(C(X, Y), \mathrm{PL}(X, Y))$ is an (s, σ)-manifold pair, where $\mathrm{PL}(X, Y)$ denotes the subspace of $C(X, Y)$ consisting of all PL maps. It is natural to conjecture that the triple

$$(C(X, Y), \mathrm{LIP}(X, Y), \mathrm{PL}(X, Y))$$

is an (s, Σ, σ)-manifold triple. In this paper, we prove this conjecture in the case where Y is an open set in Euclidean space $\mathbf{R}^r$ (cf. [SW_1, Conjecture 3.3]).

Main result. *Let X be a non-zero dimensional compact polyhedron and Y an open set in Euclidean space $\mathbf{R}^r$ $(r > 0)$. Then the triple*

$$(C(X, Y), \mathrm{LIP}(X, Y), \mathrm{PL}(X, Y))$$

is an (s, Σ, σ)-manifold triple.

This is obtained as a corollary of the following:

0.1. **Theorem.** *Let X be a non-zero dimensional compact polyhedron and Y a non-degenerate locally compact convex set in Euclidean space $\mathbf{R}^r$. Then*

$$(C(X, Y), \mathrm{LIP}(X, Y), \mathrm{PL}(X, Y)) \cong (s, \Sigma, \sigma).$$

1. Preliminaries

A closed set A in a metric space $W = (W, d)$ is a *Z-set* if for each map $f: Q \to W$ and each $\varepsilon > 0$, there is a map $g: Q \to W \backslash A$ with $d(f, g) < \varepsilon$. A countable union of Z-sets in W is called a *Z_σ-set.* A subset M of W is a *cap* (or *fd-cap*) *set* for W if $M = \bigcup_{n \in \mathbf{N}} M_n$, where $M_1 \subset M_2 \subset \cdots$ is a tower of (resp. finite-dimensional) compact Z-sets in W with the property

(fd-)cap: For each (finite-dimensional) compact set A in W, each $\varepsilon > 0$ and each $m \in \mathbf{N}$, there is an $n \geq m$ $(\in \mathbf{N})$ and an embedding $h: A \to M_n$ such that $h|A \cap M_m = \mathrm{id}$ and $d(h, \mathrm{id}) < \varepsilon$.

or (fd-)cap sets, refer to [Ch].

The following characterization of (s, Σ, σ)-(or (Q, Σ, σ)-)manifold triples is given in [SW$_2$].

1.1. **Theorem.** *A triple (W, M, N) of spaces is an (s, Σ, σ)-(or (Q, Σ, σ)-) manifold triple if and only if W is an s-(or Q-)manifold, and there is a tower $M_1 \subset M_2 \subset \cdots$ of compact Q-manifolds in W with* **cap** *such that $M = \bigcup_{i \in \mathbf{N}} M_i$, each M_i is a Z-set in M_{i+1} and each $M_i \cap N$ is an fd-cap set for M_i.*

A map $f: X \to Y$ between metric spaces is *locally Lipschitz* (abbreviated *LIP*) if each $x \in X$ has a neighborhood U such that $f|U$ is Lipschitz. Notice, each LIP map $f: X \to Y$ is Lipschitz if X is compact. A metric space Y is an *absolute LIP extensor* (abbreviated *ALE*) if every LIP map $f: A \to Y$ of a closed set of an arbitrary metric space X has a LIP extension $\tilde{f}: X \to Y$.

1.2. **Lemma.** *Any compact convex set Y in Euclidean space $\mathbf{R}^r$ is an ALE.*

Proof. By [Wo, Lemma 4], the nearest point retraction $\nu: \mathbf{R}^r \to Y$ defined by

$$\|x - \nu(x)\| = \mathrm{dist}(x, Y) = \inf\{\|x - y\| \,|\, y \in Y\}$$

is Lipschitz, where $\|\cdot\|$ is the Euclidean norm. Since $\mathbf{R}^r$ is an ALE by [LV, Theorem 5.7], Y is also an ALE. □

2. Lipschitz constants of PL maps

For each $x \in \mathbf{R}^n$, $x(i)$ denotes the i-th coordinate of x. The Euclidean norm of $\mathbf{R}^n$ is Lipschitz equivalent to the norm defined as follows:

$$\|x\| = |x(1)| + \cdots + |x(n)| \qquad \text{for } x \in \mathbf{R}^n.$$

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_q\}$ be the canonical orthonormal basis for $\mathbf{R}^q$, that is, $\mathbf{e}_i(j) = 0$ if $i \neq j$ and $\mathbf{e}_i(i) = 1$. The q-th symmetric group is denoted by Σ_q. For each $\alpha = (\alpha(1), \cdots, \alpha(q)) \in \Sigma_q$, $\Delta(\alpha)$ denotes the full simplicial complex with vertices

$$v_0^\alpha = 0, \qquad v_1^\alpha = \mathbf{e}_{\alpha(1)},$$
$$v_2^\alpha = \mathbf{e}_{\alpha(1)} + \mathbf{e}_{\alpha(2)}, \cdots, \qquad v_q^\alpha = \mathbf{e}_{\alpha(1)} + \cdots + \mathbf{e}_{\alpha(q)}.$$

2.1. **Lemma.** *For each linear (affine) map* $f\colon |\Delta(\alpha)| \to \mathbf{R}$,

$$\operatorname{lip} f = \max\{|f(v_i^\alpha) - f(v_{i-1}^\alpha)| \,|\, i = 1, \dots, q\}.$$

Proof. First observe that for each $x \in |\Delta(\alpha)|$,

$$1 \geq x(\alpha(1)) \geq x(\alpha(2)) \geq \cdots \geq x(\alpha(q)) \geq 0$$

and

$$x = (1 - x(\alpha(1))) \cdot v_0^\alpha + (x(\alpha(1)) - x(\alpha(2))) \cdot v_1^\alpha + \cdots \\ + (x(\alpha(q-1)) - x(\alpha(q))) \cdot v_{q-1}^\alpha + x(\alpha(q)) \cdot v_q^\alpha.$$

Since f is linear, it follows that

$$f(x) = (1 - x(\alpha(1))) \cdot f(v_0^\alpha) + (x(\alpha(1)) - x(\alpha(2))) \cdot f(v_1^\alpha) + \cdots \\ + (x(\alpha(q-1)) - x(\alpha(q))) \cdot f(v_{q-1}^\alpha) + x(\alpha(q)) \cdot f(v_q^\alpha).$$

Then for each x, $x' \in |\Delta(\alpha)|$,

$$\begin{aligned} |f(x) - f(x')| &\leq |x(\alpha(1)) - x'(\alpha(1))| \cdot |f(v_1^\alpha) - f(v_0^\alpha)| + \cdots \\ &\quad + |x(\alpha(q)) - x'(\alpha(q))| \cdot |f(v_q^\alpha) - f(v_{q-1}^\alpha)| \\ &\leq \|x - x'\| \cdot \max\{|f(v_i^\alpha) - f(v_{i-1}^\alpha)| \,|\, i = 1, \dots, q\}. \end{aligned}$$

Thus we have the result. □

For each $c > 0$ and $v \in \mathbf{R}^q$, let $h_{c,v}\colon \mathbf{R}^q \to \mathbf{R}^q$ be the (affine) homeomorphism defined by $h_{c,v}(x) = c \cdot x + v$. For each $p \in \mathbf{N}$, we define the triangulation $K(q,p)$ of $I^q = [0,1]^q$ as follows:

$$K(q,p) = \{h_{1/p,v}(\sigma) | \sigma \in \Delta(\alpha), \alpha \in \Sigma_q, v \in V_p^q\},$$

where

$$V_p^q = \left\{0, \frac{1}{p}, \dots, \frac{p-1}{p}\right\}^q \subset I^q.$$

2.2. **Lemma.** *Let* $f\colon I^q \to \mathbf{R}$ *be a map such that* $f|\sigma$ *is linear for each* $\sigma \in K(q,p)$. *Then*

$$\operatorname{lip} f = \operatorname{lip}(f|K(q,p)^0).$$

Proof. Let $k = \operatorname{lip}(f|K(q,p)^0)$. It suffices to show that $\operatorname{lip} f \leq k$. First we show that $\operatorname{lip}(f|\sigma) \leq k$ for each $\sigma \in K(q,p)$. Choose an $\alpha \in \Sigma_q$ and $v \in V_p^q$ so that $h_{1/p,v}^{-1}(\sigma) \in \Delta(\alpha)$. Observe that $\operatorname{lip}(h_{1/p,v}^{-1}) = p$ and

$$\|h_{1/p,v}(v_i^\alpha) - h_{1/p,v}(v_{i-1}^\alpha)\| = \frac{1}{p}$$

for each $i = 1, \dots, q$. By Lemma 2.1, we have

$$\begin{aligned} \operatorname{lip}(f|\sigma) &\leq \operatorname{lip}(f \circ h_{1/p,v} | |\Delta(\alpha)|) \cdot \operatorname{lip}(h_{1/p,v}^{-1}|\sigma) \\ &\leq p \cdot \max\{|fh_{1/p,v}(v_i^\alpha) - fh_{1/p,v}(v_{i-1}^\alpha)| \,|\, i = 1, \dots, q\} \\ &\leq p \cdot \max\{k \cdot \|h_{1/p,v}(v_i^\alpha) - h_{1/p,v}(v_{i-1}^\alpha)\| \,|\, i = 1, \dots, q\} \\ &= k. \end{aligned}$$

Now we show $\operatorname{lip} f \leq k$. Let $x \neq x' \in I^q = |K(q,p)|$. If x and x' are contained in same $\sigma \in K(q,p)$, then

$$|f(x) - f(x')| \leq \operatorname{lip}(f|\sigma) \cdot \|x - x'\| \leq k \cdot \|x - x'\|.$$

Otherwise choose points $x_0, x_1, \ldots, x_n$ on the straight line segment connecting x and x' so that $x_0 = x$, $x_n = x'$, and each pair x_{i-1} and x_i are contained in some $\sigma_i \in K(q,p)$. Since

$$\|x - x'\| = \|x_0 - x_1\| + \|x_1 - x_2\| + \cdots + \|x_{n-1} - x_n\|,$$

we have

$$\begin{aligned} |f(x) - f(x')| &\leq |f(x_0) - f(x_1)| + \cdots + |f(x_{n-1}) - f(x_n)| \\ &\leq \operatorname{lip}(f|\sigma_1) \cdot \|x_0 - x_1\| + \cdots + \operatorname{lip}(f|\sigma_n) \cdot \|x_{n-1} - x_n\| \\ &\leq k \cdot \|x - x'\|. \end{aligned}$$

This completes the proof. □

2.3. **Corollary.** *Let $f: I^q \to \mathbf{R}^r$ be a map such that $f|\sigma$ is linear for each $\sigma \in K(q,p)$. Then*

$$\operatorname{lip} f \leq r \cdot \operatorname{lip}(f|K(q,p)^0).$$

Proof. For each $i = 1, \ldots, r$, let $\pi_i: \mathbf{R}^r \to \mathbf{R}$ denote the projection onto the i-th coordinates. By Lemma 2.2,

$$\begin{aligned} \operatorname{lip} \pi_i f &= \operatorname{lip}(\pi_i f|K(q,p)^0) \\ &\leq (\operatorname{lip} \pi_i) \cdot (\operatorname{lip}(f|K(q,p)^0)) \\ &\leq \operatorname{lip}(f|K(q,p)^0). \end{aligned}$$

Then for each x, $x' \in I^q$,

$$\begin{aligned} \|f(x) - f(x')\| &= |\pi_1 f(x) - \pi_1 f(x')| + \cdots + |\pi_r f(x) - \pi_r f(x')| \\ &\leq (\operatorname{lip} \pi_1 f + \cdots + \operatorname{lip} \pi_r f) \cdot \|x - x'\| \\ &\leq r \cdot \operatorname{lip}(f|K(q,p)^0) \cdot \|x - x'\|. \quad \square \end{aligned}$$

2.4. **Lemma.** *Let Y be a convex set in Euclidean space $\mathbf{R}^r$. Then for each $q \in \mathbf{N}$, $k > 0$ and $\varepsilon > 0$, there is a map*

$$\varphi: k\text{-LIP}(I^q, Y) \to \mathrm{PL}(I^q, Y) \cap rk\text{-LIP}(I^q, Y)$$

which is ε-close to id.

Proof. Choose $p \in \mathbf{N}$ so large that

$$\operatorname{mesh} K(q,p) < \varepsilon/(r+1)k.$$

Since Y is convex, the desired map φ can be defined as follows:

$$\varphi(f)|K(q,p)^0 = f|K(q,p)^0$$

and

$$\varphi(f)|\sigma \text{ is linear for each } \sigma \in K(q,p).$$

In fact, $\operatorname{lip}\varphi(f) \le r \cdot \operatorname{lip} f \le rk$ by the above corollary. For each $x \in I^q$, we have a $v \in K(q, p)^0$ such that $\|x - v\| < \varepsilon/(r+1)k$. Then

$$\begin{aligned}\|\varphi(f)(x) - f(x)\| &\le \|\varphi(f)(x) - \varphi(f)(v)\| + \|f(x) - f(v)\| \\ &\le rk \cdot \|x - v\| + k \cdot \|x - v\| \\ &\le \varepsilon.\end{aligned}$$

Thus φ is ε-close to id. $\square$

3. Space of k-Lipschitz PL maps

Let X be a non-zero dimensional compact polyhedron in Euclidean space $\mathbf{R}^q$ and Y a non-degenerate locally compact convex set in Euclidean space $\mathbf{R}^r$. Then we can use the metrics for X and Y defined by the norm in §2 instead of Euclidean norm to prove the theorem. This norm is of great advantage to our problem as seen in Section 2 and since the map $\|\cdot\|: \mathbf{R}^q \to \mathbf{R}$ is PL. Let us regard $C(X, Y)$ as a subset of the Banach space $C(X, \mathbf{R}^r)$ with the sup-norm

$$\|f\| = \sup\{\|f(x)\| \mid x \in X\}.$$

Then note that $\mathrm{PL}(X, Y) \subset C(X, Y)$ and each k-$\mathrm{LIP}(X, Y)$ are convex sets in $C(X, \mathbf{R}^r)$.

3.1. **Lemma.** *For each $k > 0$, $\mathrm{PL}(X, Y) \cap k$-$\mathrm{LIP}(X, Y)$ is infinite dimensional.*

Proof. Without loss of generality, we may assume $0 \in Y$. Let K be a triangulation of X and $v_0 \in K^0$ be a non-isolated vertex. For each $n \in \mathbf{N}$, choose any $v_n \in \mathrm{Lk}(v_0, \mathrm{Sd}^n K)^0$, where $\mathrm{Sd}^n K$ is the n-th barycentric subdivision of K. Let $g_n \in \mathrm{PL}(X, Y)$ so that $g_n(v) = 0$ for $v \in (\mathrm{Sd}^n K)^0 \backslash \{v_n\}$, $g_n(v_n) \ne 0$ and $g_n|\sigma$ is linear for each $\sigma \in \mathrm{Sd}^n K$. By choosing $g_n(v_n)$ close to 0, we have

$$g_n \in \mathrm{PL}(X, Y) \cap k\text{-}\mathrm{LIP}(X, Y).$$

It is easy to verify that $\{g_n\}_{n\in\mathbf{N}}$ are linearly independent. Thus $\mathrm{PL}(X, Y) \cap k$-$\mathrm{LIP}(X, Y)$ is infinite-dimensional. $\square$

By [Ge$_2$, Lemma 4.3], $\mathrm{PL}(X, Y)$ is σ-fd-compact, that is, a countable union of finite-dimensional compacta. Then $\mathrm{PL}(X, Y) \cap k$-$\mathrm{LIP}(X, Y)$ is an infinite-dimensional σ-fd-compact convex set in the Banach space $C(X, \mathbf{R}^r)$. If Y is compact,

$$\begin{aligned}&\mathrm{cl}_{C(X, \mathbf{R}^r)}(\mathrm{PL}(X, Y) \cap k\text{-}\mathrm{LIP}(X, Y)) \\ &\quad = \mathrm{cl}_{C(X, Y)}(\mathrm{PL}(X, Y) \cap k\text{-}\mathrm{LIP}(X, Y)) \subset k\text{-}\mathrm{LIP}(X, Y),\end{aligned}$$

which are compact by Arzela-Ascoli's Theorem. By [Do, Theorem 2], we have the following theorem.

3.2. **Theorem.** *If Y is compact, then, for each $k > 0$,*

$$(\mathrm{cl}_{C(X,Y)}(\mathrm{PL}(X,Y) \cap k\text{-}\mathrm{LIP}(X,Y)), \mathrm{PL}(X,Y) \cap k\text{-}\mathrm{LIP}(X,Y)) \cong (Q,\sigma). \quad \square$$

In the above, it is natural to ask whether or not

$$\mathrm{cl}_{C(X,Y)}(\mathrm{PL}(X,Y) \cap k\text{-}\mathrm{LIP}(X,Y)) = k\text{-}\mathrm{LIP}(X,Y).$$

The author has not succeeded in giving its proof nor a counterexample. This is related to Problem 4.1.

The following lemma follows from the proof of [SW$_1$, Lemma 1.2]:

3.3. **Lemma.** *For each $k > 0$ and $\varepsilon > 0$, there exists a map*

$$\psi: k\text{-}\mathrm{LIP}(X,Y) \to k\text{-}\mathrm{LIP}(X,Y) \setminus \mathrm{LIP}_k(X,Y)$$

such that ψ is ε-close to id *and*

$$\psi(\mathrm{PL}(X,Y) \cap k\text{-}\mathrm{LIP}(X,Y)) \subset \mathrm{PL}(X,Y) \cap k\text{-}\mathrm{LIP}(X,Y).$$

Proof. In fact, we use the norm $\|\cdot\|$ in §2 which is PL. Then it is easily observed that the map ψ defined in [SW$_1$, Lemma 1.2] maps $\mathrm{PL}(X,Y) \cap k\text{-}\mathrm{LIP}(X,Y)$ into itself. $\square$

3.4. **Lemma.** *For each compact set $A \subset C(X,Y)$ and $\varepsilon > 0$, there is a $k > 0$ and a map*

$$\varphi: A \to \mathrm{PL}(X,Y) \cap k\text{-}\mathrm{LIP}(X,Y)$$

which is ε-close to id.

Proof. Without loss of generality, we can assume that $X \subset I^q$. Let $R: C(I^q, Y) \to C(X,Y)$ be the restriction map, that is, $R(f) = f|X$ for each $f \in C(I^q, Y)$. Note that for any $k > 0$,

$$R(\mathrm{PL}(I^q,Y) \cap k\text{-}\mathrm{LIP}(I^q,Y)) \subset \mathrm{PL}(X,Y) \cap k\text{-}\mathrm{LIP}(X,Y).$$

Since Y is an AR, there is an extension map $E: C(X,Y) \to C(I^q,Y)$, namely a right inverse of R, i.e., $R \circ E = \mathrm{id}$. In fact, the evaluation $e: X \times C(X,Y) \to Y$ extends to a map $\tilde{e}: I^q \times C(X,Y) \to Y$. Then E can be defined by $E(f)(x) = \tilde{e}(x,f)$. By [SW$_1$, Lemma 1.4], we have a $k' > 0$ and a map $\varphi': E(A) \to k'\text{-}\mathrm{LIP}(I^q,Y)$ which is $\varepsilon/2$-close to id. Let $k = rk' > 0$. By Lemma 2.4, we have a map

$$\psi: k'\text{-}\mathrm{LIP}(I^q,Y) \to \mathrm{PL}(I^q,Y) \cap k\text{-}\mathrm{LIP}(I^q,Y)$$

which is $\varepsilon/2$-close to id. Then the composition

$$\varphi = R \circ \psi \circ \varphi' \circ E: A \to \mathrm{PL}(X,Y) \cap k\text{-}\mathrm{LIP}(X,Y)$$

is the desired map. In fact, for each $f \in A$ and $x \in X$,

$$\begin{aligned}
\|\varphi(f)(x) - f(x)\| &= \|\psi(\varphi'(E(f)))(x) - E(f)(x)\| \\
&\le \|\psi(\varphi'(E(f)))(x) - \varphi'(E(f))(x)\| \\
&\quad + \|\varphi'(E(f))(x) - E(f)(x)\| \\
&\le \|\psi - \mathrm{id}\| + \|\varphi' - \mathrm{id}\| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \square
\end{aligned}$$

4. Proof of Theorem 0.1

First recall that $C(X, Y) \cong s$. Write $Y = \bigcup_{n\in\mathbf{N}} Y_n$, where each Y_n is a non-degenerate compact convex set and $Y_n \subset \operatorname{int}_Y Y_{n+1}$. For each $n \in \mathbf{N}$, let

$$M_n = \operatorname{cl}_{C(X,Y)}(\mathrm{PL}(X, Y_n) \cap n\text{-}\mathrm{LIP}(X, Y_n)).$$

It follows from Lemma 3.3 that each M_n is a Z-set in M_{n+1}. And by Theorem 3.2, $M_n \cong Q$ and

$$\mathrm{PL}(X, Y) \cap M_n = \mathrm{PL}(X, Y) \cap n\text{-}\mathrm{LIP}(X, Y_n)$$

is an fd-cap set for M_n.

We will show that $\mathrm{LIP}(X, Y) = \bigcup_{n\in\mathbf{N}} M_n$. Without loss of generality, we can assume that $X \subset I^q$. Let $f \in \mathrm{LIP}(X, Y)$. From compactness, $f(X) \subset Y_{n'}$ for some $n' \in \mathbf{N}$. By Lemma 1.2, f has an extension $\tilde{f} \in \mathrm{LIP}(I^q, Y_{n'})$. Let $n \geq \max\{n', r \cdot \operatorname{lip} \tilde{f}\}$. Then by Lemma 2.4,

$$\tilde{f} \in \operatorname{cl}_{C(I^q,Y)} \mathrm{PL}(I^q, Y_n) \cap n\text{-}\mathrm{LIP}(I^q, Y_n).$$

Therefore $f = \tilde{f}|X \in M_n$.

By using Lemma 3.4, we can prove similarly as [SW$_1$, Theorem 2.1] that the tower $\{M_n\}_{n\in\mathbf{N}}$ has the property **cap** for $C(X, Y)$. Thus Theorem 0.1 follows from Theorem 1.1. □

Finally we pose the following problem:

4.1. **Problem.** If Y is a non-degenerate compact convex set, then for each $k > 0$, is

$$(k\text{-}\mathrm{LIP}(X, Y), \mathrm{LIP}_k(X, Y), \mathrm{PL}(X, Y) \cap \mathrm{LIP}_k(X, Y)) \cong (Q, \Sigma, \sigma)?$$

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