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ON THE MODULUS OF CONE ABSOLUTELY SUMMING OPERATORS AND VECTOR MEASURES OF BOUNDED VARIATION

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ABSTRACT. Let E and F be Banach lattices. It is shown that if F has the Levi and the Fatou property, then the ordered Banach space $\mathscr{L}^l(E,F)$ of cone absolutely summing operators is a Banach lattice and an order ideal of the Riesz space $\mathscr{L}^r(E,F)$ of regular operators. The same argument yields a Jordan decomposition of F-valued vector measures of bounded variation.

Throughout the paper E and F denote Banach lattices and $\mathscr{F}$ an algebra of subsets of some set Ω. Identify F with the canonical image of F into its bidual F^{**}. F is said to have property (P), if there exists a positive contractive projection $F^{**} \to F$. If F has property (P), by Schlotterbeck's theorem [5] the Banach space $\mathscr{L}^l(E,F)$ of cone absolutely summing operators is a Banach lattice and an order ideal of the Riesz space $\mathscr{L}^r(E,F)$ of all regular operators $E \to F$. Using this result Schmidt generalizes in [6] the Jordan decomposition theorems of Diestel, Faires and Morrison [2, 3]. He proved that if F has property (P), then the ordered Banach space $bva(\mathscr{F},F)$ of all vector measures $\mathscr{F} \to F$ having bounded variation is a Banach lattice and an order ideal of the Riesz space $oba(\mathscr{F},F)$ of all order bounded vector measures $\mathscr{F} \to F$.

The main purpose of this note is to give a short proof of Schlotterbeck's theorem. The proof is based on the notion of a Nakano space and avoids duality arguments. The same type of proof yields Schmidt's theorem on vector measures of bounded variation without representing them by cone absolutely summing operators on the space of step functions.

Let us now fix some notions. A Banach lattice F is said to satisfy the Levi property (=weak Fatou property in [8]) if every increasing norm bounded net of F^+ has a supremum. Note that the Levi property implies Dedekind completeness. The norm of a Banach lattice F is a Fatou norm if $0 \leq v_\tau \uparrow v \in$

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F implies $\|v_\tau\| \uparrow \|v\|$. F is called a Nakano space [1] if it satisfies the Levi property and if in addition its norm is Fatou.

Any dual Banach lattice is a Nakano space [1, Theorems 19.12 and 19.13] and has property (P) [4, p. 299]. Moreover, the class of Nakano spaces includes all Banach lattices having property (P). Indeed, since F^{**} is a Nakano space, a norm bounded net $0 \leq v_\tau \uparrow$ of $F \subset F^{**}$ has a supremum v^{**} in F^{**} satisfying $\|v^{**}\| = \sup \|v_\tau\|$, and it follows easily that any positive contractive projection $P: F^{**} \to F$ satisfies $v_\tau \uparrow Pv^{**} \in F$ and $\|Pv^{**}\| = \sup \|v_\tau\|$. No example seems to be known of a Nakano space which does not have property (P). As is well known, a Banach lattice F with separating order continuous dual F_n^* is a Nakano space if and only if it satisfies property (P), this holds if and only if F is a perfect Banach lattice. Similarly, Dedekind complete AM-spaces with strong unit are the only AM-spaces having property (P) as well as the only Nakano AM-spaces.

We shall now state the theorem.

Theorem 1. *Let E, F be Banach lattices and suppose F to be a Nakano space. Then $\mathscr{L}^l(E, F)$ is a Banach lattice and an order ideal of $\mathscr{L}^r(E, F)$.*

Proof. Let $T \in \mathscr{L}^l(E, F)$, $u \in E^+$, and let $\pi(u)$ denote the collection of all finite families $\{u_1, \dots, u_n\} \subset E^+$ satisfying $u = \sum u_i$. Since the set

$$D = \left\{ \sum_i |Tu_i| : \{u_1, \dots, u_n\} \in \pi(u) \right\}$$

is directed upward [8, p. 122] and norm bounded (T is cone absolutely summing), by the Levi property there exists $y = \sup D$. According to [7, Lemma 2.1] $T \in \mathscr{L}^r(E, F)$ and $y = |T|u$. Moreover, by Fatou property we get

$$\begin{aligned} \sum_i \| |T|u_i \| &= \sum_i \left\| \sup_{\pi(u_i)} \sum_j |Tu_{ij}| \right\| \\ &\leq \sum_i \left(\sup_{\pi(u_i)} \sum_j \|Tu_{ij}\| \right) \leq \sup_{\pi(u)} \sum \|Tu_{ij}\| \leq \|T\|_l \|u\|, \end{aligned}$$

thus $|T| \in \mathscr{L}^l(E, F)$ and $\| |T| \|_l \leq \|T\|_l$. It follows that $\mathscr{L}^l(E, F)$ is an ideal of $\mathscr{L}^r(E, F)$, and since obviously $\|T\|_l \leq \| |T| \|_l$, the l-norm is a Riesz norm, and the proof is complete.

Remarks. 1. The proof combined with the duality principle [7, Lemma 2.2] and the duality of l- and m-norms [4, Theorem IV.3.8] yields a simplified proof of Schlotterbeck's theorem on the regularity of majorized operators [5].

2. The same type of proof is applicable to vector measures and yields the following Schmidt's decomposition theorem [6, Theorem 4.2].

Theorem 2. *If F is a Nakano space, then $bva(\mathscr{F}, F)$ is a Banach lattice and an order ideal of $oba(\mathscr{F}, F)$.*

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