MONOTONICITY IN TERMS OF ORDER OF THE ZEROS
OF THE DERIVATIVES OF BESSEL FUNCTIONS

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ABSTRACT. An elementary Sturm technique is shown to provide an alternative and simpler proof of the result that the known monotonicity of the zeros of fixed rank of the Bessel function of the first kind implies monotonicity for the zeros of its derivative for orders between -1 and 0. The reasoning applies to other Bessel functions.

The chief objective is to show that Sturm methods suffice to infer the monotonicity in ν, $-1 < \nu < 0$, of $j'_{\nu k}$ from that of $j_{\nu k}$, $k = 1, 2, \ldots$.

Only minor changes are required to show more directly than in [3] but also by Sturm methods that $j'_{\nu k}$ increases when $\nu > 0$ and so this is recorded as well. For the same reason a result for the Bessel function of the second kind, $Y_\nu(x)_\nu$, $\nu > -1/2$, is also noted. Similar remarks apply to other Bessel functions.

There are Sturm-type proofs for the monotonicity of $j_{\nu k}$, $\nu > 0$, [1, 3] and also for $j'_{\nu k}$, $\nu > 0$, [3] but none such have been offered for $-1 < \nu < 0$. For this interval, monotonicity has been established otherwise both for $j_{\nu k}$ [2; 6, p. 509] and for $j'_{\nu k}$ [4].

As usual, $j_{\nu k}$ and $j'_{\nu k}$ denote the respective k-th positive zeros of $J_\nu(x)$ and its derivative, $k = 1, 2, \ldots$, except that $j'_{01} = 0$ to assure continuity as $\nu \to 0+$. This is the Bessel function of first kind and order ν. It satisfies the differential equation.

$$(1) \qquad (xy'_\nu)' + x^{-1}(x^2 - \nu^2)y_\nu = 0, \qquad y_\nu = J_\nu(x).$$

Use will be made of the function

$$W(x) = J_\mu(x)[xJ'_\nu(x)] - J_\nu(x)[xJ'_\mu(x)]$$

and its derivative which, in view of (1), can be expressed as

$$xW'(x) = (\nu^2 - \mu^2)J_\nu(x)J_\mu(x).$$

Required also are the inequalities

(2) $\qquad j_{\nu k} < j_{\mu,k+1}, \qquad -1 < \mu < \nu \le \mu + 1, \quad k = 1, 2, \ldots .$

These follow from the interlacing relations in [6, p. 479] since $j_{\nu k} \le j_{\mu+1,k}$.

Theorem A [4]. *For each fixed* $k = 1, 2, 3, \ldots$, *the function* $j'_{\nu k}$ *increases for* $-1 < \nu < 0$.

Proof. It is required to establish that

(3) $\qquad j'_{\mu k} < j'_{\nu k}, \qquad -1 < \mu < \nu < 0, \quad k = 1, 2, \ldots .$

Here $\mu^2 > \nu^2$. Hence

$$W'(x) < 0, \qquad j_{\nu k} < x < j_{\mu,k+1}.$$

Moreover, $W(j_{\mu,k+1}) = -j_{\mu,k+1} J_\nu(j_{\mu,k+1}) J'_\mu(j_{\mu,k+1}) > 0$.
 Therefore, $W(x) > 0$, $j_{\nu k} < x < j_{\nu,k+1}$.

If $j'_{\nu k} \ge j_{\mu,k+1}$, the conclusion would be obvious. In case $j_{\nu k} < j'_{\nu k} < j_{\mu,k+1}$, then $W(j'_{\nu k}) > 0$. Thus,

$$0 < -j'_{\nu k} J_\nu(j'_{\nu k}) J'_\mu(j'_{\nu k}),$$

so that $(-1)^{k+1} J'_\mu(j'_{\nu k}) > 0$, since $(-1)^k J_\nu(j'_{\nu k}) > 0$, for $-1 < \nu < 0$.

This establishes that $j'_{\nu k} > j'_{\mu k}$, as asserted.

Theorem B [5, p. 248; 6, p. 510]. *For each fixed* $k = 1, 2, 3, \ldots$, *the function* $j'_{\nu k}$ *increases,* $\nu > 0$.

Proof. With $\nu > \mu > 0$, it will be shown that $j'_{\nu k} > j'_{\mu k}$. Without restricting generality, ν may be taken less than $\mu + 1$, so that (2) holds. Here $\nu^2 > \mu^2$ and so

$$W'(x) > 0, \qquad j_{\nu,k-1} < x < j_{\mu k}, \qquad k = 1, 2, \ldots,$$

where $j_{\nu 0} = 0$ so as to include the case $j'_{\nu 1}$; here $J_\nu(0) = J_\mu(0) = 0$.

Now, $W(j_{\nu,k-1}) = j_{\nu,k-1} J_\mu(j_{\nu,k-1}) J'_\nu(j\nu, k-1) \ge 0$, $k = 1, 2, \ldots$. Hence, $W(x) > 0$, $j_{\nu,k-1} < x < j_{\mu k}$. For $j'_{\nu k} \ge j_{\mu k}$ the conclusion is obvious. For $j_{\nu,k-1} < j'_{\nu k} < j_{\mu k}$, it follows that $W(j'_{\nu k}) > 0$, whence $J_\nu(j'_{\nu k}) J'_\mu(j'_{\nu k}) < 0$. Thus, $(-1)^{k+1} J'_\mu(j'_{\nu k}) > 0$ so that $j'_{\nu k} > j'_{\mu k}$, as asserted.

Remark. The same line of reasoning allows one to infer that $y'_{\nu k}$ increases, $\nu > -1/2$, from the known result [6, p. 509] that $y_{\nu k}$ increases for $\nu > -1/2$, for fixed $k = 1, 2, \ldots$. As usual $y_{\nu k}$, $y'_{\nu k}$ denote the respective kth positive zeros of the Bessel function $Y_\nu(x)$ and its derivative.

In carrying out the details, it must be remembered both that $y_{\mu k} < y_{\nu k} < y_{\mu+1,k}$, $-1/2 < \mu < \nu$, with $\nu < \mu + 1$, and that

(4) $\qquad y_{\mu+1,k} < y_{\mu,k+1}, \qquad \mu > -\tfrac{1}{2}, \quad k = 1, 2, \ldots,$

as is implicit in [6, §15.22, pp. 479–480].

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