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A NOTE ON TONG PAPER: THE ASYMPTOTIC BEHAVIOR OF A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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ABSTRACT. In this paper we point out an error in paper [2] and study the asymptotic behavior of the differential equation

$$L_n x + f(t,x) = r(t).$$

The results obtained are extensions of some of the results of [2].

1. INTRODUCTION

Recently, Tong [2] considered the equation

$$u'' + f(t,u) = 0, \tag{1}$$

where $f\colon [0,\infty)\times \mathbf{R} = (-\infty,\infty) \to \mathbf{R}$ is a continuous function, and proved the following theorem.

Theorem $(*)$. *Assume that there are two nonnegative continuous functions $v(t)$, $\phi(t)$ on $[0,\infty)$ and a continuous function $g(u)$, for $u \ge 0$, such that*

(i) $\int^{\infty} v(t)\phi(t)\,dt < \infty$;
(ii) *for* $u > 0$, $g(u)$ *is positive and nondecreasing*;
(iii) $|f(t,u)| < v(t)\phi(t)g(|u|/t)$, *for* $t \ge 1$, $-\infty < u < \infty$.

Then every solution $u(t)$ of (1) *satisfies $u'(t) = O(1)$ as $t\to\infty$, and* (1) *has solutions which are asymptotic to $a + bt$, where a, b are constants and $b \ne 0$.*

The purpose of this note is to point out an error in [2] and study the asymptotic behavior for a larger class of solutions of the equation

$$L_n x + f(t,x) = r(t), \tag{2}$$

where $n \ge 2$ and L_n denotes the disconjugate differential operator

$$L_n = \frac{1}{p_n(t)}\frac{d}{dt}\frac{1}{p_{n-1}(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{p_1(t)}\frac{d}{dt}\frac{\bullet}{p_0(t)}. \tag{3}$$

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We assume that p_i, $r\colon [0,\infty) \to \mathbf{R}$ and $f\colon [0,\infty) \times \mathbf{R} \to \mathbf{R}$ are continuous with $p_i(t) > 0$, $0 \le i \le n$. Put

$$L_0 x(t) = \frac{x(t)}{p_0(t)}, \tag{4}$$
$$L_i x(t) = \frac{1}{p_i(t)} \frac{d}{dt} L_{i-1} x(t), \qquad 1 \le i \le n,$$

and let $i_k \in \{1, 2, \ldots, n-1\}$, t, $s \in [0,\infty)$ and

$$I_0 = 1, \tag{5}$$
$$I_k(t, s; p_{i_k}, \ldots, p_{i_1}) = \int_s^t p_{i_k}(r) I_{k-1}(r, s; p_{i_{k-1}}, \ldots, p_{i_1})\, dr.$$

It is easily verified that

$$I_k(t, s; p_{i_k}, \ldots, p_{i_1}) = \int_s^t p_{i_1}(r) I_{k-1}(t, r; p_{i_k}, \ldots, p_{i_2})\, dr. \tag{6}$$

For convenience of notation we let

$$J_{n-1}(t, s) = p_0(t) I_{n-1}(t, s; p_1, \ldots, p_{n-1}), J_{n-1}(t) = J_{n-1}(t, 0). \tag{7}$$

2. Main result

We point out an error in Theorem $(*)$; for example, consider the equation

$$u'' - \frac{2}{t^4} u^2 = 0 \quad \text{for } t \ge 1 \tag{8}$$

Let $v(t) = t^{-4}$, $\phi(t) = t^2$, and $g(u) = u^2$. Then conditions (i)–(iii) are satisfied, but Equation (8) has a solution $u(t) = t^2$ that does not satisfy $u'(t) = O(1)$ as $t \to \infty$.

Theorem. *Suppose that $\int^\infty p_i(t)\, dt = \infty$, $1 \le i \le n-1$, and that there are two nonnegative continuous functions $v(t)$, $\phi(t)$, for $t \ge 0$, and a continuous function $g(x)$ for $x \ge 0$ such that*

(i) $\int^\infty p_n(t) v(t) \phi(t)\, dt < \infty$, $\int^\infty p_n(t) |r(t)|\, dt < \infty$;
(ii) *for $x > 0$, $g(x)$ is positive nondecreasing and* $\int_1^\infty \frac{du}{g(u)} = \infty$;
(iii) $|f(t, x)| \le v(t) \phi(t) g(|x| / J_{n-1}(t))$ *for* $t \ge 0$, $x \in \mathbf{R}$.

Then every solution $x(t)$ of Equation (2) satisfies $x(t) = O(J_{n-1}(t))$ as $t \to \infty$, and $L_{n-1} x(t) = O(1)$ as $t \to \infty$.

Remark. If $n = 2$ and $p_i(t) = 1$, for $i = 0, 1, 2$, $r(t) = 0$, then Equation (2) reduces to (1); by our theorem we obtain Theorem $(*)$.

Proof of the theorem. Let $x(t)$ be a solution of (2) defined on $[t_0, \infty)$, $t_0 > 0$. Integrating (2) n times on $[t_0, t]$ gives

$$\begin{aligned} \frac{x(t)}{p_0(t)} &= \sum_{i=0}^{n-1} C_i I_i(t, t_0; p_1, \dots, p_i) \\ &\quad + \int_{t_0}^{t} I_{n-1}(t, s; p_1, \dots, p_{n-1}) p_n(s)[r(s) - f(s, x(s))]\, ds, \end{aligned} \tag{9}$$

where c_i, $0 \le i \le n-1$, are constants. Noting that $\int^{\infty} p_i(t)\, dt = \infty$, $1 \le i \le n-1$, we have

$$\lim_{t\to\infty} \frac{I_i(t, t_0; p_1, \dots, p_i)}{I_{n-1}(t, t_0; p_1, \dots, p_{n-1})} = 0, \quad 0 \le i \le n-2.$$

From (9) we obtain

$$|x(t)| \le J_{n-1}(t) \left[C + \int_{t_0}^{t} p_n(r) v(r) \phi(\gamma) g\left(\frac{|x(r)|}{J_{n-1}(r)} \right) dr \right],$$

where $C > 0$ is a constant. By Bihari's [1] inequality, we have

$$\frac{|x(t)|}{J_{n-1}(t)} \le G^{-1}\left(G(C) + \int_{t_0}^{t} v(r)\phi(\gamma) p_n(r)\, dr \right).$$

Here $G(x) = \int_1^x \frac{dt}{g(t)}$ and $G^{-1}(x)$ is the inverse function of $G(x)$. From the fact $g(t) > 0$ we know that $G(x)$ is increasing; hence, $G^{-1}(x)$ exists and is also increasing.

Now let $M = G(C) + \int_{t_0}^{\infty} v(t)\phi(t) p_n(t)\, dt$; since $G^{-1}(x)$ is increasing, we have

$$\frac{|x(t)|}{J_{n-1}(t)} \le G^{-1}(M). \tag{10}$$

From (2) it follows that

$$L_{n-1}x(t) = L_{n-1}x(t_0) + \int_{t_0}^{t} p_n(s)(r(s) - f(s, x(s)))\, ds.$$

By conditions (i), (ii), (iii) and (10), we have

$$\begin{aligned} \int_{t_0}^{t} p_n(s)|f(s, x(s))|\, ds &\le \int_{t_0}^{t} p_n(s) v(s)\phi(s) g\left(\frac{|x(s)|}{J_{n-1}(s)} \right) ds \\ &\le g(G^{-1}(M)) \int_{t_0}^{\infty} p_n(s) v(s) \phi(s)\, ds < \infty. \end{aligned}$$

Therefore $L_{n-1}x(t) = O(1)$, as $t \to \infty$. □

References

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