PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 2, February 1990

INTEGRAL GROUP RINGS WITH TRIVIAL CENTRAL UNITS

JÜRGEN RITTER AND SUDARSHAN K. SEHGAL

(Communicated by Maurice Auslander)

ABSTRACT. In this note finite groups G whose integral group ring $\mathbf{Z}G$ has only trivial central units are classified.

In this note we classify finite groups G whose integral group ring $\mathbf{Z}G$ has only trivial central units; a unit being trivial if it is of the form $\pm g$, $g \in G$. This question was raised by Goodaire and Parmenter [2].

It was proved by Higman ([1], [3, p. 57]) that *all* units of $\mathbf{Z}G$ are trivial if and only if

(a) G is Abelian with exponent a divisor of 4 or 6, or
(b) $G = K_8 \times E$, where K_8 is the quaternion group of order 8 and E is an elementary Abelian 2-group.

It follows easily that all units of a commutative group ring $\mathbf{Z}G$ are trivial if and only if:

For every $x \in G$ and every natural number j, relatively prime to $|G|$, we have $x^j = x$ or $x^j = x^{-1}$.

Denoting by $\sim$ conjugation in G, we state our result:

Theorem. *Let G be a finite group. All central units of $\mathbf{Z}G$ are trivial if and only if for every $x \in G$ and every natural number j, relatively prime to $|G|$, $x^j \sim x$ or $x^j \sim x^{-1}$.*

Proof. At first we recall that any finite group of central units of $\mathbf{Z}G$ consists of trivial units only [4, p. 46]. It suffices to prove that the following conditions are equivalent:

(1) $\mathbf{Z}G$ has only a finite number of central units.
(2) The character field $\mathbf{Q}(\chi)$ of each absolutely irreducible character χ of G is either $\mathbf{Q}$ or imaginary quadratic.
(3) For every $x \in G$ and every natural number j, relatively prime to $|G|$, $x^j \sim x$ or $x^j \sim x^{-1}$.

Received by the editors April 10, 1989 and, in revised form, May 19, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 16A25, 16A26; Secondary 20C05.

This work is supported by NSERC Grant A-5300.

(a) We shall first show that (1) and (2) are equivalent. Since the center of $\mathbf{Q}G$ is generated by the class sums, the center Z of $\mathbf{Z}G$ is an order in the center of $\mathbf{Q}G$, the latter being the direct sum of all character fields $\mathbf{Q}(\chi)$ [3, p. 544]. Hence Z is of finite additive index in the unique maximal order $\oplus_\chi O_\chi$ of $\oplus_\chi \mathbf{Q}(\chi)$, with O_χ denoting the ring of integers in $\mathbf{Q}(\chi)$. Thus the unit group of Z is of finite index in the multiplicative group $\oplus_\chi (O_\chi)^\times$ [4, p. 49]. It follows that (1) holds precisely when $(O_\chi)^\times$ is finite for all χ which by the Dirichlet unit theorem is the same as (2).

(b) We next prove that (3) implies (2). Let σ be an automorphism of $\mathbf{Q}(\chi)/\mathbf{Q}$. Extend σ to an automorphism $\zeta \to \zeta^j$ of $\mathbf{Q}(\zeta)$ where ζ is a $|G|$th root of unity. Then $\chi^\sigma(g) = \chi(g^j) = \chi(g)$ or $\chi(g^{-1})$ by (3). We have $\chi^\sigma(g) = \chi(g)$ or $\bar{\chi}(g)$. Since $^-$ commutes with σ, it follows that $\chi + \bar{\chi} = \chi^\sigma + \bar{\chi}^\sigma$. Thus $\chi^\sigma = \chi$ or $\bar{\chi}$ by the linear independence of irreducible characters. This implies that $\mathbf{Q}(\chi) = \mathbf{Q}$ or an imaginary quadratic field.

(c) We finally show that (2) implies (3). The proof is dual of (b). For each $g \in G$ we define a function from the irreducible characters to the complex numbers, $T(g)\colon \operatorname{Irr}(G) \to \mathbf{C}$ by $\chi \to \chi(g)$. It follows by the orthogonality relations that these functions, one for each conjugacy class of G, are linearly independent. Now let $(j, |G|) = 1$. Then we have an automorphism $\zeta \to \zeta^j$ of $\mathbf{Q}(\zeta)$ where ζ is a $|G|$th root of unity. Let σ be the restriction of this automorphism to $\mathbf{Q}(\chi)$. Then $\chi^\sigma(g) = \chi(g)$ or $\chi(g^{-1})$ by (2). Thus $T(g^j) + T(g^{-j}) = T(g) + T(g^{-1})$. It follows due to the linear independence of these functions that $T(g^j) = T(g)$ or $T(g^{-1})$. Thus g^j is conjugate to g or g^{-1} as desired.

An easy consequence of (2) is:

Corollary. *If all central units of* $\mathbf{Z}G$ *are trivial then the same is true for* $\mathbf{Z}\overline{G}$, $\overline{G}$ *a homomorphic image of* G.

Examples. We close with a few examples of groups satisfying the condition of the theorem.

(a) $G = S_n$ the symmetric group on n-letters. In this case all the character fields are rational [3, p. 538].
(b) G a group of order 27. In this case, all character fields are $\mathbf{Q}$, $\mathbf{Q}(w)$, $w^3 = 1$.
(c) $G = \langle x, y \colon x^7 = 1 = y^3, x^y = x^2 \rangle$. In this case,

$$\mathbf{Q}G = \mathbf{Q} \oplus \mathbf{Q}(w) \oplus \mathbf{Q}(\sqrt{-7})_{3\times 3}, \qquad w^3 = 1.$$

Observe that $V = \mathbf{Q}(\sqrt{-7})$ is the field of index 3 in $\mathbf{Q}(\zeta)$, $\zeta^7 = 1$, and that V is a three-dimensional space over $\mathbf{Q}(\sqrt{-7})$ on which G acts irreducibly by letting x act as multiplication by ζ and y by the automorphism $\zeta \to \zeta^2$. In this example, the character fields are $\mathbf{Q}$, $\mathbf{Q}(w)$ and $\mathbf{Q}(\sqrt{-7})$.

Acknowledgment

We thank the referee for shortening the proof of the theorem.

References

1. G. Higman, *The units of group rings*, Proc. London Math. Soc. (2) **46** (1940), 231–248.
2. E. G. Goodaire and M. M. Parmenter, *Units in alternative loop rings*, Israel J. Math. **53** (1986), 209–216.
3. B. Huppert, *Endliche Gruppen* I, Springer, Berlin-Heidelberg-New York, 1967.
4. S. K. Sehgal, *Topics in group rings*, Marcel Dekker, New York, 1978.

Institut für Mathematik der Universität Augsburg, D-8900 Augsburg, West Germany

Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1 Canada