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# INTEGRAL GROUP RINGS WITH TRIVIAL CENTRAL UNITS

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ABSTRACT. In this note finite groups $G$ whose integral group ring $\mathbf{Z}G$ has only trivial central units are classified.

In this note we classify finite groups $G$ whose integral group ring $\mathbf{Z}G$ has only trivial central units; a unit being trivial if it is of the form $\pm g$, $g \in G$. This question was raised by Goodaire and Parmenter [2].

It was proved by Higman ([1], [3, p. 57]) that *all* units of $\mathbf{Z}G$ are trivial if and only if

(a) $G$ is Abelian with exponent a divisor of 4 or 6, or
(b) $G = K_8 \times E$, where $K_8$ is the quaternion group of order 8 and $E$ is an elementary Abelian 2-group.

It follows easily that all units of a commutative group ring $\mathbf{Z}G$ are trivial if and only if:

For every $x \in G$ and every natural number $j$, relatively prime to $|G|$, we have $x^j = x$ or $x^j = x^{-1}$.

Denoting by $\sim$ conjugation in $G$, we state our result:

**Theorem.** *Let $G$ be a finite group. All central units of $\mathbf{Z}G$ are trivial if and only if for every $x \in G$ and every natural number $j$, relatively prime to $|G|$, $x^j \sim x$ or $x^j \sim x^{-1}$.*

*Proof.* At first we recall that any finite group of central units of $\mathbf{Z}G$ consists of trivial units only [4, p. 46]. It suffices to prove that the following conditions are equivalent:

(1) $\mathbf{Z}G$ has only a finite number of central units.
(2) The character field $\mathbf{Q}(\chi)$ of each absolutely irreducible character $\chi$ of $G$ is either $\mathbf{Q}$ or imaginary quadratic.
(3) For every $x \in G$ and every natural number $j$, relatively prime to $|G|$, $x^j \sim x$ or $x^j \sim x^{-1}$.

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(a) We shall first show that (1) and (2) are equivalent. Since the center of $\mathbf{Q}G$ is generated by the class sums, the center $Z$ of $\mathbf{Z}G$ is an order in the center of $\mathbf{Q}G$, the latter being the direct sum of all character fields $\mathbf{Q}(\chi)$ [3, p. 544]. Hence $Z$ is of finite additive index in the unique maximal order $\oplus_\chi O_\chi$ of $\oplus_\chi \mathbf{Q}(\chi)$, with $O_\chi$ denoting the ring of integers in $\mathbf{Q}(\chi)$. Thus the unit group of $Z$ is of finite index in the multiplicative group $\oplus_\chi (O_\chi)^\times$ [4, p. 49]. It follows that (1) holds precisely when $(O_\chi)^\times$ is finite for all $\chi$ which by the Dirichlet unit theorem is the same as (2).

(b) We next prove that (3) implies (2). Let $\sigma$ be an automorphism of $\mathbf{Q}(\chi)/\mathbf{Q}$. Extend $\sigma$ to an automorphism $\zeta \to \zeta^j$ of $\mathbf{Q}(\zeta)$ where $\zeta$ is a $|G|$th root of unity. Then $\chi^\sigma(g) = \chi(g^j) = \chi(g)$ or $\chi(g^{-1})$ by (3). We have $\chi^\sigma(g) = \chi(g)$ or $\bar{\chi}(g)$. Since $^-$ commutes with $\sigma$, it follows that $\chi + \bar{\chi} = \chi^\sigma + \bar{\chi}^\sigma$. Thus $\chi^\sigma = \chi$ or $\bar{\chi}$ by the linear independence of irreducible characters. This implies that $\mathbf{Q}(\chi) = \mathbf{Q}$ or an imaginary quadratic field.

(c) We finally show that (2) implies (3). The proof is dual of (b). For each $g \in G$ we define a function from the irreducible characters to the complex numbers, $T(g)\colon \mathrm{Irr}(G) \to \mathbf{C}$ by $\chi \to \chi(g)$. It follows by the orthogonality relations that these functions, one for each conjugacy class of $G$, are linearly independent. Now let $(j, |G|) = 1$. Then we have an automorphism $\zeta \to \zeta^j$ of $\mathbf{Q}(\zeta)$ where $\zeta$ is a $|G|$th root of unity. Let $\sigma$ be the restriction of this automorphism to $\mathbf{Q}(\chi)$. Then $\chi^\sigma(g) = \chi(g)$ or $\chi(g^{-1})$ by (2). Thus $T(g^j) + T(g^{-j}) = T(g) + T(g^{-1})$. It follows due to the linear independence of these functions that $T(g^j) = T(g)$ or $T(g^{-1})$. Thus $g^j$ is conjugate to $g$ or $g^{-1}$ as desired.

An easy consequence of (2) is:

**Corollary.** *If all central units of* $\mathbf{Z}G$ *are trivial then the same is true for* $\mathbf{Z}\overline{G}$, $\overline{G}$ *a homomorphic image of* $G$.

**Examples.** We close with a few examples of groups satisfying the condition of the theorem.

(a) $G = S_n$ the symmetric group on $n$-letters. In this case all the character fields are rational [3, p. 538].
(b) $G$ a group of order 27. In this case, all character fields are $\mathbf{Q}$, $\mathbf{Q}(w)$, $w^3 = 1$.
(c) $G = \langle x, y\colon x^7 = 1 = y^3, x^y = x^2\rangle$. In this case,

$$\mathbf{Q}G = \mathbf{Q} \oplus \mathbf{Q}(w) \oplus \mathbf{Q}(\sqrt{-7})_{3\times 3}, \qquad w^3 = 1.$$

Observe that $V = \mathbf{Q}(\sqrt{-7})$ is the field of index 3 in $\mathbf{Q}(\zeta)$, $\zeta^7 = 1$, and that $V$ is a three-dimensional space over $\mathbf{Q}(\sqrt{-7})$ on which $G$ acts irreducibly by letting $x$ act as multiplication by $\zeta$ and $y$ by the automorphism $\zeta \to \zeta^2$. In this example, the character fields are $\mathbf{Q}$, $\mathbf{Q}(w)$ and $\mathbf{Q}(\sqrt{-7})$.

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