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DIRICHLET-FINITE ANALYTIC AND HARMONIC FUNCTIONS ARE BMO

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ABSTRACT. Based on a result of F. John, an elementary proof is given of the fact that Dirichlet-finite analytic and Dirichlet-finite harmonic functions are of bounded mean oscillation in the unit disk.

1. In [5] Metzger proved the rather surprising result that the space of Dirichlet-finite analytic functions on a hyperbolic Riemann surface belong to the space BMO. Subsequently, in [4], Kusunoki and Taniguchi found that the same result holds for Dirichlet-finite harmonic functions on Riemann surfaces of finite type. In this note we give an elementary proof, based on a result of John [2], that in the unit disk Dirichlet-finite analytic and harmonic functions are BMO.

2. Denote the unit disk by $U : |z| < 1$, and

$$AD(U) = \left\{ f \in A(U) \colon D_U(f) = \int_U\!\!\int |f'(z)|^2 \, dx\, dy < \infty \right\},$$
$$HD(U) = \left\{ u \in H(U) \colon D_U(u) = \int_U\!\!\int |\operatorname{grad} u|^2 \, dx\, dy < \infty \right\},$$

as the spaces of Dirichlet-finite analytic, and Dirichlet-finite harmonic, functions, respectively.

Let $\mathrm{BMO}A(U)$ be the space of analytic functions on U which belong to the Hardy class $H^2(U)$ and satisfy

$$\sup_{\zeta\in U} \int_U\!\!\int |f'(z)|^2 \log\left|\frac{1-\bar{\zeta}z}{z-\zeta}\right| dx\, dy < \infty,$$

and define $\mathrm{BMO}H(U)$ analogously for harmonic functions, replacing $|f'|^2$ by $|\operatorname{grad} u|^2$.

A sufficient condition for a real-valued differentiable function u to belong to $\mathrm{BMO}(U)$ in the sense of John–Nirenberg [3] has been given by John [2], namely: $\sup_{z\in U}(1-|z|)|\operatorname{grad} u(z)| < \infty$.

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We can now readily establish the following:

Theorem. $AD(U) \subseteq \mathrm{BMO}A(U)$.

Proof. Firstly, $AD(U) \subseteq H^2(U)$ (cf. Heins [1]). Take $f \in AD(U)$, $f = u + iv$. By the areal mean value property applied to $\mathrm{Re}[(f')^2]$, $\mathrm{Im}[(f')^2]$,

$$(f'(z))^2 = \frac{1}{\pi\rho^2}\int_0^\rho \int_0^{2\pi} [f'(z + re^{i\theta})]^2 r\,dr\,d\theta,$$

where $\rho = 1 - |z|$. Then

$$|\operatorname{grad} u(z)|^2 = |f'(z)|^2 \leq \frac{1}{\pi(1-|z|)^2}\int_U\int |f'(z)|^2\,dx\,dy = \frac{1}{\pi(1-|z|)^2}D_U(f).$$

Therefore,

$$\sup_{z\in U}(1-|z|)|\operatorname{grad} u(z)| \leq \sqrt{\frac{1}{\pi}D_U(f)} < \infty.$$

Since $u \in h^2(U)$, the John result implies $u \in \mathrm{BMO}H(U)$, and hence $f \in \mathrm{BMO}A(U)$.

For any function $u \in HD(U)$, $u \in \mathrm{Re}\, f$ for some $f \in AD(U)$, and the preceding proof yields:

Corollary. $HD(U) \subseteq \mathrm{BMO}H(U)$.

References

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