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COMPACT COMPOSITION OPERATORS ON L^1

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ABSTRACT. The composition operator induced by a holomorphic self-map of the unit disc is compact on L^1 of the unit circle if and only if it is compact on the Hardy space H^2 of the disc. This answers a question posed by Donald Sarason: it proves that Sarason's integral condition characterizing compactness on L^1 is equivalent to the asymptotic condition on the Nevanlinna counting function which characterizes compactness on H^2.

INTRODUCTION

We answer a question posed by Donald Sarason ([9], §II) concerning the compactness of operators induced on L^1 of the unit circle by composing Poisson integrals with holomorphic self-maps of the unit disc. Sarason represented such operators as integral operators, and gave a necessary and sufficient condition, in terms of the integral kernel, for compactness. He observed that compactness of such a (holomorphic) *composition operator* on L^1 implies its compactness on the Hardy space H^2, and asked if the converse is true. The purpose of this note is to show that this is indeed the case:

- ♦ *A holomorphic composition operator is compact on* L^1 *if and only if it is compact on* H^2.

We prove this result in §4, after reformulating the problem in §§1 and 2, and setting out some preparatory material on compactness in §3.

A consequence of our result is the equivalence of Sarason's integral condition characterizing the compact composition operators on L^1 ([9], Propositions 2 and 3), and the first author's asymptotic condition on the Nevanlinna counting function, which characterizes compactness on H^2 ([11], Theorem 2.3).

A crucial role was played in [11] by a special case of C. S. Stanton's remarkable formula for integral means [3], [13], [14]. The ideas of [11], along with the

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general version of Stanton's formula, provide the key to the main result of this paper.

1. **Harmonic Hardy spaces.** For the material in this section we refer the reader to the books of Duren ([2], Chapter 1), and Rudin ([8], Chapter 11). Let D denote the unit disc of the complex plane. For $1 \leq p \leq \infty$, let L^p denote the usual complex Lebesgue space on the unit circle ∂D, taken with respect to normalized Lebesgue measure σ. We denote the Poisson integral of $f \in L^p$ by $P[f]$, so that $P[f]$ is a harmonic function on D whose radial limit coincides with f a.e. on ∂D. It is well known that for $1 < p < \infty$ the map

$$f \to P[f] \tag{1}$$

establishes an isometric isomorphism taking L^p onto the "harmonic Hardy space" h^p: those complex harmonic functions u on D which satisfy the growth condition

$$\|u\|_p^p := \sup_{0 \leq r < 1} \int_{\partial D} |u(r\zeta)|^p \, d\sigma(\zeta) < \infty . \tag{2}$$

The same is true for the space h^∞ of bounded harmonic functions on D, taken in the supremum norm. However for $p = 1$ the situation is more subtle. Here it is the space M of complex Borel measures, with the variation norm, that the Poisson integral sends isometrically onto h^1, with L^1 mapped onto the closure in h^1 of the harmonic polynomials (i.e. the closure of the Poisson integrals of the trigonometric polynomials).

The usual Hardy space H^p is the closed subspace of h^p that consists of holomorphic functions.

2. **Composition operators.** In this paper, the symbol b always denotes a holomorphic self-map of D, i.e. a member of the unit ball of H^∞. Each such map induces a linear *composition operator* on the space of harmonic functions on D by means of the formula:

$$C_b u = u \circ b \qquad (u \text{ harmonic on } D).$$

Littlewood's Subordination Principal ([6]; see also [2], Chapter 1, Theorem 1.7, page 10) asserts that if $b(0) = 0$, $1 \leq p < \infty$, and $u \in h^p$, then $\|C_b u\|_p \leq \|u\|_p$, i.e. that the operator C_b is a contraction on h^p. Clearly the same is true for h^∞ without the restriction on $b(0)$. Since conformal automorphisms of D induce Banach space isomorphisms of h^p, it follows from Littlewood's principle that for $1 \leq p \leq \infty$:

- ♦ *Each composition operator C_b induces a bounded linear operator on h^p.*

Since the composition of holomorphic function is again holomorphic, it follows that each composition operator acts boundedly on the ordinary (holomorphic) Hardy spaces of D, and it is in this context that they have received the most study (see [1] for a survey of recent developments in this setting). Thanks to the isomorphisms discussed in §1, the operator C_b can be viewed as acting boundedly on the Banach spaces L^p or M. Since C_b preserves the closure in h^1 of the harmonic polynomials, it also induces a bounded operator on L^1. In this context, Sarason identified C_b as an integral operator, and employed Schur's boundedness test for integral operators on L^p spaces to give another proof of Littlewood's principle ([9], Proposition 1).

3. **Compact composition operators.** It has been known for a while that compactness of a composition operator on one H^p space ($p < \infty$) implies compactness on all of them ([10], Theorem 6.1). Thus in treating the compactness problem for Hardy spaces, one need only concentrate on the Hilbert space H^2. Recently the compact composition operators on H^2 were characterized as follows ([11], Theorem 2.3):

♦ C_b is compact on $H^2 \Leftrightarrow \lim_{|w|\to 1-} \frac{N_b(w)}{1-|w|} = 0$,

where $N_b(w)$ is the Nevanlinna counting function for b:

$$N_b(w) = \sum_{z\in b^{-1}\{w\}} \log\frac{1}{|z|} \qquad (w \in \mathbf{C}\setminus\{b(0)\}),$$

which we understand to take the value zero whenever $w \notin b(D)$.

Suppose $1 < p < \infty$. Then the Riesz projection theorem asserts that h^p can be written as a Banach space direct sum of H^p and the space of complex conjugates of H^p functions that vanish at the origin. If $b(0) = 0$ then C_b preserves both summands hence is compact on h^p if and only if it is compact on H^p. Once this is said, the automorphism argument indicated in §2 renders the extra condition $b(0) = 0$ irrelevant. Thus for $1 < p < \infty$ the compactness problems for composition operators on both holomorphic and harmonic Hardy spaces are the same.

If $p = 1$, however, the Riesz projection theorem no longer operates: in fact H^1 is no longer a direct summand of h^1 ([7]; see also [5], Chapter 9, p. 154). Nevertheless, H^1 is still a closed subspace of h^1; so compactness of C_b on h^1 implies compactness on H^1, and therefore on H^2. Furthermore, Sarason observed that compactness on M (a.k.a. h^1) is equivalent to compactness on L^1, and he asked if compactness on H^2 implies compactness on h^1 ([9], §II). In the next section we answer Sarason's question in the affirmative.

4. **Main theorem.** *For a holomorphic self-map b of D, the following conditions are equivalent*:

(a) *C_b is compact on h^1.*
(b) *C_b is compact on H^2 (and so on H^p for all $p < \infty$).*
(c) $\lim_{|w|\to 1-} \frac{N_b(w)}{1-|w|} = 0$.

Proof. The implications (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) have already been discussed, so only (c) $\Rightarrow$ (a) needs to be proved. As in [11], the key to success lies in Stanton's formula, which we take a few paragraphs to discuss.

Recall that every function g that is subharmonic on D has a *Riesz mass* $\mu[g]$: a positive Borel measure with support in the closed unit disc, defined by the distributional formula

$$\mu[g] = \frac{1}{2\pi}\Delta g,$$

where Δ denotes the Laplacian (see [4], Chapter 3). In plain English, for every *test function* (infinitely differentiable function on $\mathbf{C}$ with compact support) τ we have

$$\int \tau \, d\mu[g] = \frac{1}{2\pi}\int g\Delta\tau \, dA,$$

where dA denotes Lebesgue area measure on the plane ([4], Lemmas 3.6 and 3.8).

We have already discussed the Nevanlinna counting function $N_b(w)$. For each $0 < r < 1$ there is also the *reduced counting function*:

$$N_b(w, r) = \sum \left\{\log\frac{r}{|z|} = z \in b^{-1}\{w\} \cap rD\right\},$$

which by convention vanishes whenever $w \notin b(rD)$. We can now state:

Stanton's Formula [3], [13], [14]. *If g is subharmonic on D, then for any holomorphic self-map b of D and any $0 < r < 1$:*

$$\int_{\partial D} g(b(r\zeta))\, d\sigma(\zeta) = g(b(0)) + \int N_b(w, r)\, d\mu[g](w). \tag{1}$$

Since $N_b(w, r)$ increases to $N_b(w)$ as r increases to 1, and since the left-hand side of (1) increases monotonically with r ([2], Theorem 1.5, p. 9), we obtain for the special case $g = |u|$, a crucial formula for the h^1 norm of a composition with b.

Corollary. *If $u \in h^1$, then*

$$\|C_b u\|_1 = |u(b(0))| + \int N_b(w)\, d\mu[|u|](w). \tag{2}$$

For the special case: $b =$ identity function on D, we have

$$N_b(w, r) = \log\frac{r}{|w|}, \qquad (|w| < r)$$

and

$$N_b(w) = \log\frac{1}{|w|}, \qquad (|w| < 1),$$

with both functions vanishing outside the indicated range of w. Along with (1) and (2) this yields the following formulas for both the actual and "reduced" norm of a function $u \in h^1$:

$$\|u\|_1 = |u(0)| + \int \log\frac{1}{|w|}\, d\mu[|u|](w), \tag{3}$$

$$\int_{\partial D} |u(r\zeta)|\, d\sigma(\zeta) = |u(0)| + \int_{rD} \log\frac{r}{|w|}\, d\mu[|u|](w), \qquad (0 < r < 1). \tag{4}$$

We can now proceed with the proof of the main theorem. There is no loss of generality in assuming that $b(0) = 0$, so we do this for the remainder of the proof. A routine normal families argument shows that the unit ball of h^1 is compact in the topology of uniform convergence on compact subsets of D, and from this it follows quickly that in order to show C_b compact it is enough to check that $\|C_b u_n\|_1 \to 0$ whenever $\{u_n\}$ is a sequence in the unit ball of h^1 that converges to zero uniformly on compact subsets of D.

So fix such a sequence $\{u_n\}$. Let $\mu_n = \mu[|u_n|]$, the Riesz mass of the subharmonic function $|u_n|$. Let $\varepsilon > 0$ be given.

We are assuming that the counting function of b satisfies the asymptotic decay condition (c) of the statement of the main theorem, so we may choose $0 < r < 1$ so that

$$N_b(w) \le \varepsilon \quad \log\frac{1}{|w|} \qquad (\text{all } r < |w| < 1). \tag{5}$$

From (2):

$$\begin{aligned}\|C_b u_n\|_1 &= |u_n(0)| + \int_{rD} N_b(w)\, d\mu_n(w) + \int_{D\setminus rD} N_b(w)\, d\mu_n(w)\\ &\le |u_n(0)| + \int_{rD} N_b(w)\, d\mu_n(w) + \varepsilon \int_D \log\frac{1}{|w|}\, d\mu_n(w),\end{aligned}$$

so from (3) and the fact that $\|u_n\|_1 \le 1$ for each n, we obtain

$$\|C_b u_n\|_1 \le (1-\varepsilon)|u_n(0)| + \varepsilon + \int_{rD} N_b(w)\, d\mu_n(w). \tag{6}$$

We turn our attention to the last integral. Here we require *Littlewood's inequality* ([10]; see also [11], p. 380), which asserts that, because $b(0) = 0$, we have

$$N_b(w) \le \log\frac{1}{|w|} \qquad (|w| \le 1). \tag{7}$$

Thus:

$$\begin{aligned}\int_{rD} N_b(w)\, d\mu_n(w) &\le \int_{rD} \log\frac{1}{|w|}\, d\mu_n(w)\\ &= \int_{rD} \log\frac{r}{|w|}\, d\mu_n(w) + \mu_n(rD)\log\frac{1}{r}\\ &= \int_{\partial D} |u_n(r\zeta)|\, d\sigma(\zeta) - |u_n(0)| + \mu_n(rD)\log\frac{1}{r},\end{aligned}$$

where the last line follows from (4). Since the sequence $\{u_n\}$ converges to zero uniformly on compact subsets of D, the first two terms in the last line above tend to zero as $n \to \infty$. Substituting all this back into (6) we get:

$$\|C_b u_n\|_1 \le \varepsilon + \mu_n(rD)\log\frac{1}{r} + o(1) \qquad (n \to \infty). \tag{8}$$

Recall that we wish to show $\|C_b u_n\|_1 \to 0$, and to this end have fixed an arbitrary positive number ε, and have chosen $r \in (0,1)$, also fixed, and depending only on ε. The desired result will follow from (8) once we verify that

$$\lim_{n\to\infty} \mu_n(rD) = 0. \tag{9}$$

To prove (9), let τ be a test function on the plane with $0 \le \tau \le 1$, support $\tau \subset ((r+1)/2)D$, and $\tau \equiv 1$ on rD. From these conditions on τ and our earlier discussion of the Riesz mass:

$$\mu_n(rD) \le \int \tau\, d\mu_n = \frac{1}{2\pi}\int \Delta\tau |u_n|\, dA \le \frac{M}{2\pi}\int_{\operatorname{spt}\tau} |u_n|\, dA, \tag{10}$$

where $M = \max\{|\Delta\tau(z)| : z \in \mathbf{C})$ (finite because $\Delta\tau$ is also a test function).

Since the sequence $\{u_n\}$ converges to zero uniformly on compact subsets of D, and the support of τ is such a set, the last integral in (10) converges to zero. This proves (9), and completes the proof of the theorem. □

Remarks. (a) Littlewood's inequality, along with (2) and (3) of the last section provide an alternate proof of Littlewood's subordination principle for the space h^1. A similar proof, along with the appropriate version of Stanton's formula, leads to the same result for H^2, and as we mentioned in the Introduction, lays the groundwork for the characterization of the compact composition operators on that space.

(b) Sarason characterizes the compactness of C_b in terms of the integral kernel

$$K_b(\zeta,\eta) = \frac{1-|b(\zeta)|}{|\eta - b(\zeta)|^2} \qquad (\zeta,\eta \in \partial D),$$

where $b(\zeta)$ denotes the radial limit of b, which exists for a.e. $\zeta \in \partial D$. His result can be rephrased as follows: *C_b is compact on h^1 if and only if*:

$$\int K_b(\zeta,\eta)\, d\sigma(\zeta) = 1 \quad \text{for all } \eta \in \partial U. \tag{$*$}$$

The result of the last section shows that condition $(*)$ is equivalent to the requirement

$$N_b(w) = o(1-|w|) \qquad \text{as } |w| \to 1-.$$

(c) Sarason also asked in [9] if an extreme point of the H^∞ unit ball could satisfy $(*)$. In view of our main result, this question can be rephrased: *Can an extreme point of the unit ball of H^∞ induce a compact operator on H^2?* In [12] we have shown that this *can* happen: in fact there is a *univalent* extreme point that induces a compact composition operator on H^2, and which therefore satisfies $(*)$.

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