

## COMPACT COMPOSITION OPERATORS ON $L^1$

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**ABSTRACT.** The composition operator induced by a holomorphic self-map of the unit disc is compact on  $L^1$  of the unit circle if and only if it is compact on the Hardy space  $H^2$  of the disc. This answers a question posed by Donald Sarason: it proves that Sarason's integral condition characterizing compactness on  $L^1$  is equivalent to the asymptotic condition on the Nevanlinna counting function which characterizes compactness on  $H^2$ .

### INTRODUCTION

We answer a question posed by Donald Sarason ([9], §II) concerning the compactness of operators induced on  $L^1$  of the unit circle by composing Poisson integrals with holomorphic self-maps of the unit disc. Sarason represented such operators as integral operators, and gave a necessary and sufficient condition, in terms of the integral kernel, for compactness. He observed that compactness of such a (holomorphic) *composition operator* on  $L^1$  implies its compactness on the Hardy space  $H^2$ , and asked if the converse is true. The purpose of this note is to show that this is indeed the case:

◆ *A holomorphic composition operator is compact on  $L^1$  if and only if it is compact on  $H^2$ .*

We prove this result in §4, after reformulating the problem in §§1 and 2, and setting out some preparatory material on compactness in §3.

A consequence of our result is the equivalence of Sarason's integral condition characterizing the compact composition operators on  $L^1$  ([9], Propositions 2 and 3), and the first author's asymptotic condition on the Nevanlinna counting function, which characterizes compactness on  $H^2$  ([11], Theorem 2.3).

A crucial role was played in [11] by a special case of C. S. Stanton's remarkable formula for integral means [3], [13], [14]. The ideas of [11], along with the

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general version of Stanton's formula, provide the key to the main result of this paper.

**1. Harmonic Hardy spaces.** For the material in this section we refer the reader to the books of Duren ([2], Chapter 1), and Rudin ([8], Chapter 11). Let  $D$  denote the unit disc of the complex plane. For  $1 \leq p \leq \infty$ , let  $L^p$  denote the usual complex Lebesgue space on the unit circle  $\partial D$ , taken with respect to normalized Lebesgue measure  $\sigma$ . We denote the Poisson integral of  $f \in L^p$  by  $P[f]$ , so that  $P[f]$  is a harmonic function on  $D$  whose radial limit coincides with  $f$  a.e. on  $\partial D$ . It is well known that for  $1 < p < \infty$  the map

$$(1) \quad f \rightarrow P[f]$$

establishes an isometric isomorphism taking  $L^p$  onto the "harmonic Hardy space"  $h^p$ : those complex harmonic functions  $u$  on  $D$  which satisfy the growth condition

$$(2) \quad \|u\|_p^p := \sup_{0 \leq r < 1} \int_{\partial D} |u(r\zeta)|^p d\sigma(\zeta) < \infty.$$

The same is true for the space  $h^\infty$  of bounded harmonic functions on  $D$ , taken in the supremum norm. However for  $p = 1$  the situation is more subtle. Here it is the space  $M$  of complex Borel measures, with the variation norm, that the Poisson integral sends isometrically onto  $h^1$ , with  $L^1$  mapped onto the closure in  $h^1$  of the harmonic polynomials (i.e. the closure of the Poisson integrals of the trigonometric polynomials).

The usual Hardy space  $H^p$  is the closed subspace of  $h^p$  that consists of holomorphic functions.

**2. Composition operators.** In this paper, the symbol  $b$  always denotes a holomorphic self-map of  $D$ , i.e. a member of the unit ball of  $H^\infty$ . Each such map induces a linear *composition operator* on the space of harmonic functions on  $D$  by means of the formula:

$$C_b u = u \circ b \quad (u \text{ harmonic on } D).$$

*Littlewood's Subordination Principle* ([6]; see also [2], Chapter 1, Theorem 1.7, page 10) asserts that if  $b(0) = 0$ ,  $1 \leq p < \infty$ , and  $u \in h^p$ , then  $\|C_b u\|_p \leq \|u\|_p$ , i.e. that the operator  $C_b$  is a contraction on  $h^p$ . Clearly the same is true for  $h^\infty$  without the restriction on  $b(0)$ . Since conformal automorphisms of  $D$  induce Banach space isomorphisms of  $h^p$ , it follows from Littlewood's principle that for  $1 \leq p \leq \infty$ :

◆ *Each composition operator  $C_b$  induces a bounded linear operator on  $h^p$ .*

Since the composition of holomorphic function is again holomorphic, it follows that each composition operator acts boundedly on the ordinary (holomorphic) Hardy spaces of  $D$ , and it is in this context that they have received the most study (see [1] for a survey of recent developments in this setting). Thanks to the isomorphisms discussed in §1, the operator  $C_b$  can be viewed as acting

boundedly on the Banach spaces  $L^p$  or  $M$ . Since  $C_b$  preserves the closure in  $h^1$  of the harmonic polynomials, it also induces a bounded operator on  $L^1$ . In this context, Sarason identified  $C_b$  as an integral operator, and employed Schur's boundedness test for integral operators on  $L^p$  spaces to give another proof of Littlewood's principle ([9], Proposition 1).

**3. Compact composition operators.** It has been known for a while that compactness of a composition operator on one  $H^p$  space ( $p < \infty$ ) implies compactness on all of them ([10], Theorem 6.1). Thus in treating the compactness problem for Hardy spaces, one need only concentrate on the Hilbert space  $H^2$ . Recently the compact composition operators on  $H^2$  were characterized as follows ([11], Theorem 2.3):

$$\blacklozenge \quad C_b \text{ is compact on } H^2 \Leftrightarrow \lim_{|w| \rightarrow 1^-} \frac{N_b(w)}{1-|w|} = 0,$$

where  $N_b(w)$  is the Nevanlinna counting function for  $b$ :

$$N_b(w) = \sum_{z \in b^{-1}\{w\}} \log \frac{1}{|z|} \quad (w \in \mathbb{C} \setminus \{b(0)\}),$$

which we understand to take the value zero whenever  $w \notin b(D)$ .

Suppose  $1 < p < \infty$ . Then the Riesz projection theorem asserts that  $h^p$  can be written as a Banach space direct sum of  $H^p$  and the space of complex conjugates of  $H^p$  functions that vanish at the origin. If  $b(0) = 0$  then  $C_b$  preserves both summands hence is compact on  $h^p$  if and only if it is compact on  $H^p$ . Once this is said, the automorphism argument indicated in §2 renders the extra condition  $b(0) = 0$  irrelevant. Thus for  $1 < p < \infty$  the compactness problems for composition operators on both holomorphic and harmonic Hardy spaces are the same.

If  $p = 1$ , however, the Riesz projection theorem no longer operates: in fact  $H^1$  is no longer a direct summand of  $h^1$  ([7]; see also [5], Chapter 9, p. 154). Nevertheless,  $H^1$  is still a closed subspace of  $h^1$ ; so compactness of  $C_b$  on  $h^1$  implies compactness on  $H^1$ , and therefore on  $H^2$ . Furthermore, Sarason observed that compactness on  $M$  (a.k.a.  $h^1$ ) is equivalent to compactness on  $L^1$ , and he asked if compactness on  $H^2$  implies compactness on  $h^1$  ([9], §II). In the next section we answer Sarason's question in the affirmative.

**4. Main theorem.** *For a holomorphic self-map  $b$  of  $D$ , the following conditions are equivalent:*

- (a)  $C_b$  is compact on  $h^1$ .
- (b)  $C_b$  is compact on  $H^2$  (and so on  $H^p$  for all  $p < \infty$ ).
- (c)  $\lim_{|w| \rightarrow 1^-} \frac{N_b(w)}{1-|w|} = 0$ .

*Proof.* The implications (a)  $\Rightarrow$  (b)  $\Leftrightarrow$  (c) have already been discussed, so only (c)  $\Rightarrow$  (a) needs to be proved. As in [11], the key to success lies in Stanton's formula, which we take a few paragraphs to discuss.

Recall that every function  $g$  that is subharmonic on  $D$  has a *Riesz mass*  $\mu[g]$ : a positive Borel measure with support in the closed unit disc, defined by the distributional formula

$$\mu[g] = \frac{1}{2\pi} \Delta g,$$

where  $\Delta$  denotes the Laplacian (see [4], Chapter 3). In plain English, for every *test function* (infinitely differentiable function on  $\mathbb{C}$  with compact support)  $\tau$  we have

$$\int \tau d\mu[g] = \frac{1}{2\pi} \int g \Delta \tau dA,$$

where  $dA$  denotes Lebesgue area measure on the plane ([4], Lemmas 3.6 and 3.8).

We have already discussed the Nevanlinna counting function  $N_b(w)$ . For each  $0 < r < 1$  there is also the *reduced counting function*:

$$N_b(w, r) = \sum \left\{ \log \frac{r}{|z|} = z \in b^{-1}\{w\} \cap rD \right\},$$

which by convention vanishes whenever  $w \notin b(rD)$ . We can now state:

**Stanton’s Formula** [3], [13], [14]. *If  $g$  is subharmonic on  $D$ , then for any holomorphic self-map  $b$  of  $D$  and any  $0 < r < 1$ :*

$$(1) \quad \int_{\partial D} g(b(r\zeta)) d\sigma(\zeta) = g(b(0)) + \int N_b(w, r) d\mu[g](w).$$

Since  $N_b(w, r)$  increases to  $N_b(w)$  as  $r$  increases to 1, and since the left-hand side of (1) increases monotonically with  $r$  ([2], Theorem 1.5, p. 9), we obtain for the special case  $g = |u|$ , a crucial formula for the  $h^1$  norm of a composition with  $b$ .

**Corollary.** *If  $u \in h^1$ , then*

$$(2) \quad \|C_b u\|_1 = |u(b(0))| + \int N_b(w) d\mu[|u|](w).$$

For the special case:  $b =$  identity function on  $D$ , we have

$$N_b(w, r) = \log \frac{r}{|w|}, \quad (|w| < r)$$

and

$$N_b(w) = \log \frac{1}{|w|}, \quad (|w| < 1),$$

with both functions vanishing outside the indicated range of  $w$ . Along with (1) and (2) this yields the following formulas for both the actual and “reduced” norm of a function  $u \in h^1$ :

$$(3) \quad \|u\|_1 = |u(0)| + \int \log \frac{1}{|w|} d\mu[|u|](w),$$

$$(4) \quad \int_{\partial D} |u(r\zeta)| d\sigma(\zeta) = |u(0)| + \int_{rD} \log \frac{r}{|w|} d\mu[|u|](w), \quad (0 < r < 1).$$

We can now proceed with the proof of the main theorem. There is no loss of generality in assuming that  $b(0) = 0$ , so we do this for the remainder of the proof. A routine normal families argument shows that the unit ball of  $h^1$  is compact in the topology of uniform convergence on compact subsets of  $D$ , and from this it follows quickly that in order to show  $C_b$  compact it is enough to check that  $\|C_b u_n\|_1 \rightarrow 0$  whenever  $\{u_n\}$  is a sequence in the unit ball of  $h^1$  that converges to zero uniformly on compact subsets of  $D$ .

So fix such a sequence  $\{u_n\}$ . Let  $\mu_n = \mu[|u_n|]$ , the Riesz mass of the subharmonic function  $|u_n|$ . Let  $\varepsilon > 0$  be given.

We are assuming that the counting function of  $b$  satisfies the asymptotic decay condition (c) of the statement of the main theorem, so we may choose  $0 < r < 1$  so that

$$(5) \quad N_b(w) \leq \varepsilon \log \frac{1}{|w|} \quad (\text{all } r < |w| < 1).$$

From (2):

$$\begin{aligned} \|C_b u_n\|_1 &= |u_n(0)| + \int_{rD} N_b(w) d\mu_n(w) + \int_{D \setminus rD} N_b(w) d\mu_n(w) \\ &\leq |u_n(0)| + \int_{rD} N_b(w) d\mu_n(w) + \varepsilon \int_D \log \frac{1}{|w|} d\mu_n(w), \end{aligned}$$

so from (3) and the fact that  $\|u_n\|_1 \leq 1$  for each  $n$ , we obtain

$$(6) \quad \|C_b u_n\|_1 \leq (1 - \varepsilon)|u_n(0)| + \varepsilon + \int_{rD} N_b(w) d\mu_n(w).$$

We turn our attention to the last integral. Here we require *Littlewood's inequality* ([10]; see also [11], p. 380), which asserts that, because  $b(0) = 0$ , we have

$$(7) \quad N_b(w) \leq \log \frac{1}{|w|} \quad (|w| \leq 1).$$

Thus:

$$\begin{aligned} \int_{rD} N_b(w) d\mu_n(w) &\leq \int_{rD} \log \frac{1}{|w|} d\mu_n(w) \\ &= \int_{rD} \log \frac{r}{|w|} d\mu_n(w) + \mu_n(rD) \log \frac{1}{r} \\ &= \int_{\partial D} |u_n(r\zeta)| d\sigma(\zeta) - |u_n(0)| + \mu_n(rD) \log \frac{1}{r}, \end{aligned}$$

where the last line follows from (4). Since the sequence  $\{u_n\}$  converges to zero uniformly on compact subsets of  $D$ , the first two terms in the last line above tend to zero as  $n \rightarrow \infty$ . Substituting all this back into (6) we get:

$$(8) \quad \|C_b u_n\|_1 \leq \varepsilon + \mu_n(rD) \log \frac{1}{r} + o(1) \quad (n \rightarrow \infty).$$

Recall that we wish to show  $\|C_b u_n\|_1 \rightarrow 0$ , and to this end have fixed an arbitrary positive number  $\varepsilon$ , and have chosen  $r \in (0, 1)$ , also fixed, and depending only on  $\varepsilon$ . The desired result will follow from (8) once we verify that

$$(9) \quad \lim_{n \rightarrow \infty} \mu_n(rD) = 0.$$

To prove (9), let  $\tau$  be a test function on the plane with  $0 \leq \tau \leq 1$ , support  $\tau \subset ((r+1)/2)D$ , and  $\tau \equiv 1$  on  $rD$ . From these conditions on  $\tau$  and our earlier discussion of the Riesz mass:

$$(10) \quad \mu_n(rD) \leq \int \tau d\mu_n = \frac{1}{2\pi} \int \Delta\tau |u_n| dA \leq \frac{M}{2\pi} \int_{\text{spt } \tau} |u_n| dA,$$

where  $M = \max\{|\Delta\tau(z)| : z \in \mathbb{C}\}$  (finite because  $\Delta\tau$  is also a test function).

Since the sequence  $\{u_n\}$  converges to zero uniformly on compact subsets of  $D$ , and the support of  $\tau$  is such a set, the last integral in (10) converges to zero. This proves (9), and completes the proof of the theorem.  $\square$

*Remarks.* (a) Littlewood's inequality, along with (2) and (3) of the last section provide an alternate proof of Littlewood's subordination principle for the space  $h^1$ . A similar proof, along with the appropriate version of Stanton's formula, leads to the same result for  $H^2$ , and as we mentioned in the Introduction, lays the groundwork for the characterization of the compact composition operators on that space.

(b) Sarason characterizes the compactness of  $C_b$  in terms of the integral kernel

$$K_b(\zeta, \eta) = \frac{1 - |b(\zeta)|}{|\eta - b(\zeta)|^2} \quad (\zeta, \eta \in \partial D),$$

where  $b(\zeta)$  denotes the radial limit of  $b$ , which exists for a.e.  $\zeta \in \partial D$ . His result can be rephrased as follows:  $C_b$  is compact on  $h^1$  if and only if:

$$(*) \quad \int K_b(\zeta, \eta) d\sigma(\zeta) = 1 \quad \text{for all } \eta \in \partial U.$$

The result of the last section shows that condition (\*) is equivalent to the requirement

$$N_b(w) = o(1 - |w|) \quad \text{as } |w| \rightarrow 1 - .$$

(c) Sarason also asked in [9] if an extreme point of the  $H^\infty$  unit ball could satisfy (\*). In view of our main result, this question can be rephrased: *Can an extreme point of the unit ball of  $H^\infty$  induce a compact operator on  $H^2$ ?* In [12] we have shown that this *can* happen: in fact there is a *univalent* extreme point that induces a compact composition operator on  $H^2$ , and which therefore satisfies (\*).

## REFERENCES

1. C. C. Cowen, *Composition operators on Hilbert spaces of analytic functions: A status report*, Proc. Sympos. Pure Math. (to appear).
2. P. L. Duren, *Theory of  $H^p$  spaces*, Academic Press, New York, 1970.

3. M. Essén, D. F. Shea, and C. S. Stanton, *A value-distribution criterion for the class  $L \log L$ , and some related questions*, Ann. Inst. Fourier (Grenoble) **35** (1985), 127–150.
4. W. K. Hayman and P. B. Kennedy, *Subharmonic functions*, Vol. 1, Academic Press, New York, 1976.
5. K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall, Englewood Cliffs, New Jersey, 1962.
6. J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc. (2) **23** (1925), 481–519.
7. D. J. Newman, *The non-existence of projections from  $L^1$  to  $H^1$* , Proc. Amer. Math. Soc. **12** (1961), 98–99.
8. W. Rudin, *Real and complex analysis*, 2nd ed., McGraw-Hill, New York, 1974.
9. D. Sarason, *Composition operators as integral operators*, in Analysis and Partial Differential Equations, a volume of papers dedicated to M. Cotlar (C. Sadosky, editor) Marcel Dekker, New York (to appear).
10. J. H. Shapiro and P. D. Taylor, *Compact, nuclear, and Hilbert-Schmidt composition operators on  $H^2$* , Indiana Univ. Math. J. **23** (1973), 471–496.
11. J. H. Shapiro, *The essential norm of a composition operator*, Ann. of Math. **125** (1987), 375–404.
12. J. H. Shapiro and C. Sundberg, *Isolation amongst the composition operators*, Pacific J. Math. (to appear).
13. C. S. Stanton, *Riesz mass and growth problems for subharmonic functions*, Thesis, University of Wisconsin, Madison, 1982.
14. ———, *Counting functions and majorization for Jensen measures*, Pacific J. Math. **125** (1986), 459–468.

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