EMBEDDING SUBSPACES OF $L_1$ INTO $l_1^N$

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Abstract. We simplify techniques of Schechtman, Bourgain, Lindenstrauss, Milman, to prove the following. If $X$ is an $n$-dimensional subspace of $L_1$, there exists a subspace $Y$ of $l_1^N$ such that $d(X,Y) \leq 1 + \varepsilon$ whenever $N \geq C K(X)^2 \varepsilon^{-2n}$, where $K(X)$ is the $K$-convexity constant of $X$, and where $C$ is a universal constant.

1. Introduction

Consider the following problem. 

(*) Given an $n$-dimensional subspace $X$ of $L_1 = L_1(0,1)$ and $\varepsilon > 0$, what is the smallest $N = N(X,\varepsilon)$ such that there is a subspace $Y$ of $l_1^N$ with $d(X,Y) \leq 1 + \varepsilon$?

This question was considered in [3] and [4] for $X = l_2^n$, and in [6] for $X = l_p^n$. A breakthrough was made in [13] by G. Schechtman, who, using empirical distributions proved that for every subspace $X$ of $L_1(0,1)$ of dimension $n$, and every $0 < \varepsilon < \frac{1}{2}$,

\[ N(X,\varepsilon) \leq C \varepsilon^{-2} \log \left( \frac{1}{\varepsilon} \right) n^2. \]

Here, as in the sequel, $C$ denotes a universal constant, that may vary at each occurrence. Schechtman’s method was refined, and combined with facts from Banach space theory by Bourgain, Lindenstrauss, Milman [1] to yield a bound of $N(X,\varepsilon)$ that is nearly linear in $n$. In particular, they proved that for any $n$-dimensional subspace $X$ of $L_1$,

\[ N(X,\varepsilon) \leq C \varepsilon^{-2} \log(n \varepsilon^{-1})(\log n)^2 n. \]

Moreover, they showed that if $1 < p \leq 2$ and if $T_p(X)$ denote the type $p$ constant of $X$, we have, for any $\tau > 0$

\[ N(X,\varepsilon) \leq c(\tau) n \varepsilon^{-2} \left( \log(T_p(X) + 1) \right)^{1/2} (p-1)^{-3-\tau} (\log(T_p(X)\varepsilon(p-1)))^{5/2+\tau}. \]
We denote by $K(X)$ the $K$-convexity constant of $X$ (see [11]), and we prove the following, where $\varepsilon_0$ is a universal constant.

**Theorem.** For any $n$-dimensional subspace $X$ of $L^1$, and $0 < \varepsilon \leq \varepsilon_0$, we have

$$N(X, \varepsilon) \leq CK(X)^2\varepsilon^{-2}n.$$  

It has been proved by Pisier [12] that for $X \subset L^1$, $n = \dim X$,

$$K(X) \leq C(\log n)^{1/2}.$$  

Thus (4) improves on (2). It is proved in [1] that

$$K(X)^2 \leq C \log(T_p(X) + 1)/(p - 1),$$  

and thus (4) improves on (3). We do not know if the factor $K(X)$ is necessary in (4).

2. The random choice procedure

Central to our approach is a random choice argument, in spirit close to the empirical distribution method, but which avoids many of the technical complications of this method. Consider a subspace $X$ of dimension $n$ of $l_1^M$. To embed $X$ into $l_1^M'$ where $M'$ is of order $M/2$ we will, for each coordinate, flip a coin and disregard the coordinate if "head" comes up. More formally, consider a sequence $\varepsilon = (\varepsilon_i)_{i \leq M}$ of Bernoulli (or Rademacher) random variables. That is, the sequence is independent identically distributed (i.i.d.) and $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$. Consider the random diagonal operator $U_\varepsilon : l_1^M \to l_1^M$ given by $U_\varepsilon((x_i)_{i \leq M}) = ((1 + \varepsilon_i)x_i)_{i \leq M}$. We will give conditions under which $U_\varepsilon$ is, with large probability, almost an isometry when restricted to $X$. We note that $U_\varepsilon(l_1^M)$ is isometric to $l_1^{M'}$, where $M' = \text{card}\{i \leq M ; \varepsilon_i = 1\}$, and, with probability $\geq \frac{1}{2}$, we have $M' \leq M/2$.

For $x \in l_1^M$, consider the random variable $Z_x = \|U_\varepsilon(x)\| - \|x\|$, and let $A = \sup_{x \in X, \|x\|_1 \leq 1} |Z_x|$. The restriction $T_\varepsilon$ of $U_\varepsilon$ to $X$ satisfies

$$\|T_\varepsilon\| \leq 1 + A, \quad \|T_\varepsilon^{-1}\| \leq 1/(1 - A),$$  

so that when $A \leq \frac{1}{2}$ we have $d(X, U_\varepsilon(X)) \leq 1 + 3A$.

For $x = (x_i)_{i \leq M}$, we have

$$Z_x = \sum_{i \leq M} |(1 + \varepsilon_i)x_i| - \sum_{i \leq M} |x_i|$$  

$$= \sum_{i \leq M} (1 + \varepsilon_i)|x_i| - \sum_{i \leq M} |x_i| = \sum_{i \leq M} \varepsilon_i|x_i|$$
so that

\[ A = \sup_{x \in X, \|x\|_1 \leq 1} \left| \sum_{i \leq M} \varepsilon_i |x_i| \right|. \]

It follows by the comparison theorem for Rademacher processes ([7], Proposition 1) that

\[ EA \leq 2E \sup_{x \in X, \|x\|_1 \leq 1} \left| \sum_{i \leq M} \varepsilon_i x_i \right|. \]

Consider a sequence \((g_i)_{i \leq N}\) of i.i.d. \(N(0,1)\) random variables. Then by the "contraction principle" as e.g. in [5], we have

\[ EA \leq CE \sup_{x \in X, \|x\|_1 \leq 1} \left| \sum_{i \leq M} g_i x_i \right|. \]

Since \(P(A \leq 3EA) \geq \frac{1}{3}\) and

\[ P(\text{card } M' \leq \frac{1}{2} \text{card } M) \geq \frac{1}{2}, \]

we have shown the following.

**Proposition 1.** There exists two universal constants \(\alpha, C\), with the following property. If \(X\) is a subspace of \(l_1^M\) and

\[ H := E \sup_{x \in X, \|x\|_1 \leq 1} \left| \sum_{i \leq M} g_i x_i \right| < \alpha \]

then there exists \(M' \leq M/2\) and a subspace \(Y\) of \(l_1^M\) such that \(d(X, Y) \leq 1 + CH\).

**Remark.** Instead of using the comparison theorem for Rademacher processes (see [8] for a very simple proof) one can use above the usual comparison results for Gaussian processes, which, however, lie quite deeper.

3. **Proof of the theorem**

In order to successfully use Proposition 1, one must ensure that

\[ H = E \sup_{x \in X, \|x\|_1 \leq 1} \left| \sum_{i \leq M} g_i x_i \right| \]

is small. Let us denote by \(\langle \cdot, \cdot \rangle\) the scalar product of \(l_2^M\). We have

\[ H = \int \sup_{x \in X, \|x\|_1 \leq 1} \langle x, y \rangle \, d\gamma_M(y) \]
where \( \gamma_M \) is the law of \( \sum_{i \leq M} g_i e_i \), and where \( (e_i)_{i \leq M} \) is the canonical basis of \( l_2^M \). Since \( \gamma_M \) is the law of \( \sum_{i \leq M} g_i y_i \) for any orthonormal basis \( l_2^M \), we see that if \( y_1, \ldots, y_n \) is an orthonormal basis of \( X \subset l_2^M \) we have

\[
H = E \sup_{x \in X, \|x\|_1 \leq 1} \left( x, \sum_{i \leq n} g_i y_i \right).
\]

The following proposition is essentially a juxtaposition of a lemma of Lewis [9] and a lemma of Davis–Milman–Tomczak [2] which were also used in [1].

**Proposition 2.** Consider a subspace \( X \) of \( l_1^M \) of dimension \( n \). Then there exists \( M' \leq 3M/2 \) and a subspace \( Y \) of \( l_1^M \) isometric to \( X \) such that

\[
E \sup_{x \in Y, \|x\| \leq 1} \left| \sum_{i \leq M} g_i x_i \right| \leq C K(X) \left( \frac{n}{M} \right)^{1/2}.
\]

**Proof.** Consider \( D = \{-1, 1\}^N \), provided with the Haar measure \( \mu \). For a Banach space \( X \), the \( K \)-convexity constant \( K(X) \) is the norm of the natural projection from \( L_2(X) = L_2(D, \mu, X) \) onto the span of the functions \( \sum e_i x_i \), where \( e_i \) is the \( i \)th coordinate function on \( D \). Thus, if \( f \in L_2(X) \), we have

\[
\left\| \sum_{i \geq 1} e_i E(e_i f) \right\|_{L_2(X)} \leq K(X) \|f\|_{L_2(X)}.
\]

It has been observed by Tomczak–Jaegermann [14] that for any probability space \( \Omega \), if \( f \in L_2(X) = L_2(\Omega, X) \), we have

\[
(4.1) \quad \left\| \sum_{i \geq 1} g_i E(g_i f) \right\|_{L_2(X)} \leq K(X) \|f\|_{L_2(X)}
\]

where \( g_i \) is i.i.d. \( N(0, 1) \) on \( \Omega \).

As observed in [1], it is an immediate consequence of a result of Lewis [9] that for each \( n \)-dimensional subspace \( X \) of \( l_1^M \), there exists a probability measure \( \nu \) on \( \{1, \ldots, M\} \), a subspace \( Y \) of \( l_1(\nu) \), isometric to \( X \), and a basis \( (\psi_j)_{j \leq n} \) of \( Y \), that is orthogonal in \( L_2(\nu) \), and that satisfies

\[
\sum_{j=1}^n \psi_j^2 = 1, \quad \|\psi_j\|_2 = n^{-1/2}.
\]

If we split each atom of \( \nu \) of mass \( a \geq 2/M \) in \( [a M/2] + 1 \) equal pieces, we can assume that each atom of \( \nu \) has mass \( \leq 2/M \), and that \( \nu \) is now supported by \( \{1, \ldots, M'\} \) where \( M' \leq 3M/2 \). Also, we can assume that \( \nu(\{k\}) > 0 \) for \( k \leq M' \).

We denote by \( \langle \cdot, \cdot \rangle \) the scalar product in \( L_2(\nu) \). There exists

\[
f \in L_\infty(Y), \quad \|f\|_{L_\infty(Y)} \leq 1,
\]

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such that

\[ \sup_{x \in Y, \|x\|_1 \leq 1} \left( \sum_{j \leq n} g_j \psi_j, x \right) = \left( \sum_{j \leq n} g_j \psi_j, f \right) = \sum_{j \leq n} \langle \psi_j, g_j f \rangle. \]

Thus

\[ E \sup_{x \in Y, \|x\|_1 \leq 1} \left( \sum_{j \leq n} g_j \psi_j, x \right) = \sum_{j \leq n} \langle \psi_j, E(g_j f) \rangle. \]

By (4.1), setting \( x_j = E(g_j f) \), we have \( \| \sum g_j x_j \|_{L^2(Y)} \leq K(Y) \| f \|_{L^2(Y)} = K(X) \). We have

\[ \left\| \sum_{j \leq n} g_j x_j \right\|_{L^2(X)}^2 = E \int \left( \sum_{j \leq n} g_j x_j(t) \right)^2 \, d\nu(t) = \int \left( \sum_{j \leq n} x_j^2(t) \right) \, d\nu(t). \]

Now, since \( \sum_{j \leq n} \psi_j^2 = 1 \)

\[ \sum_{j \leq n} \langle \psi_j, x_j \rangle = \int \sum_{j \leq n} \psi_j(t) x_j(t) \, d\nu(t) \leq \int \left( \sum_{j \leq n} x_j^2(t) \right)^{1/2} \, d\nu(t) \leq \left\| \sum_{j \leq n} g_j x_j \right\|_{L^2(X)} \leq K(X). \]

For \( k \leq M' \), let

\[ a_k = \nu(\{k\}), \quad v_k = 1_{\{k\}}, \]

so that \((a_k^{-1/2} v_k)_{k \leq M'}\) is an orthonormal basis of \( L^2(\nu) \). As observed earlier, we have

\[ E \sup_{x \in Y, \|x\|_1 \leq 1} \left( \sum_{k \leq M'} g_k a_k^{-1/2} v_k, x \right) = E \sup_{x \in Y, \|x\|_1 \leq 1} \left( \sum_{j \leq n} g_j \psi_j, x \right) \leq n^{1/2} K(X). \]

Consider now the isometry \( T : L_1(\nu) \to l^M_1 \) given by \( T(x) = \sum_{k \leq M'} e_k (v_k, x), \) where \((e_k)\) denotes the canonical basis of \( l^M_1 \). We have

\[ E \sup_{x \in T(Y), \|x\|_1 \leq 1} \left( \sum_{k \leq M'} g_k e_k, x \right) = E \sup_{x \in Y, \|x\|_1 \leq 1} \left( \sum_{k \leq M'} g_k v_k, x \right) \]

\[ = E \sup_{x \in Y, \|x\|_1 \leq 1} \left( \sum_{k \leq M'} g_k \frac{v_k}{\sqrt{a_k}}, \sqrt{a_k}, x \right) \leq (\max a_k^{1/2}) E \sup_{x \in Y, \|x\|_1 \leq 1} \left( \sum_{k \leq M'} g_k a_k^{-1/2} v_k, x \right) \leq C(n/N)^{1/2} K(X) \]

by contraction and since \( a_k \leq 2/N \) for \( k \leq M' \). The proof is complete.
Successive applications of Propositions 2 and 1 yield the following:

**Proposition 3.** There exists two universal constants \( \alpha \), \( C \), such that if \( X \) is a subspace of \( l_1^M \) of dimension \( n \), for which \( K(X)(n/M)^{1/2} \leq \alpha \), there exists \( M' \leq 3M/4 \) and a subspace \( Y \) of \( l_1^{M'} \) for which \( d(X,Y) \leq 1 + CK(X)(n/M)^{1/2} \).

To prove the theorem, we observe that given \( \varepsilon > 0 \), a subspace \( X \) of \( L_1 \) of finite dimension is always at distance \(< 1 + \varepsilon \) of a subspace \( l_1^M \) for some \( M \) (however large). This can be seen in a number of ways, e.g. using the fact that \( \| f - E^n(f) \|_1 \to 0 \) where \( E^n \) denotes the conditional expectation with respect to the \( n \)th dyadic subalgebra. Thus, it suffices to prove the theorem for subspaces of \( l_1^N \). In that case it follows from repeated applications of Proposition 3, once we observe that \( K(Y) \leq d(X,Y)K(X) \).

### 4. Further comments

Also of interest is the problem, studied in particular in [1], of embedding a subspace \( X \) of \( L_p(p > 1) \) of dimension \( n \) in \( l_1^N \) for small \( N \). Using the random choice procedure described in §2, one is led to consider the random variable

\[
Z_x = \| U_x(x) \|^p - \| x \|^p = \sum_{i \leq M} e_i |x_i|^p
\]

and \( A = \sup_{x \in X, \| x \|_p \leq 1} |Z_x| \). To apply the method of Proposition 1, we need to have information on the Gaussian process \( V_x = \sum_{i \leq M} g_i |x_i|^p \). If one tries to deduce this information from information on the process \( W_x = \sum_{i \leq M} g_i x_i \) using the standard comparison theorems one does not get the correct result, as these comparison theorems do not take into account that \( X \) is a subspace of small dimension. Thus we seem to have no choice other than evaluating the expectation of \( A \) using the canonical distance \( \delta(x,y) = \sum_{i \leq M} (|x_i|^p - |y_i|^p)^{1/2} \) and entropy methods, i.e. Dudley's theorem as e.g. in [10]. For that purpose we observe e.g. that for

\[
\| x \|_p \leq 1, \quad \| y \|_p \leq 1,
\]

we have

\[
\delta(x,y) \leq C\| x - y \|_p/(2-p) \leq C\| x - y \|_\infty^{p/2} \quad \text{if} \quad 1 \leq p \leq 2
\]

\[
\delta(x,y) \leq C p (\text{Max}(\| x \|_\infty, \| y \|_\infty))^{(p-2)/2} \| x - y \|_\infty \quad \text{if} \quad p \geq 2
\]

and we use the entropy evaluations of [1].

While our approach in that case replaces (once the hard work on entropy evaluation has been done!) the computations of [1], Theorem 7.3 and Theorem 7.4 by an application of Dudley's theorem, it does not even decrease the power...
of the $\log n$ term there, and thus it does not yield any real improvement. A similar comment applies to the results of [1], §8.

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