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L^{∞}-BMO BOUNDEDNESS FOR A SINGULAR INTEGRAL OPERATOR

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ABSTRACT. If $K(x) = \Omega(x)/|x|^n$ is a Calderón–Zygmund kernel and $b(|x|)$ is a bounded radial function, we find conditions on b such that the singular operator whose kernel is $b(x)K(x)$ is bounded from $L^{\infty}(R^n)$ to $\mathrm{BMO}(R^n)$, or equivalently from $H^1(R^n)$ into $L^1(R^n)$.

Let $K(x) = \Omega(x)/|x|^n$ be a Calderón–Zygmund kernel, where

$$\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0,$$

and $\Omega(\lambda x) = \Omega(x)$ for $\lambda > 0$, $x \in R^n$. Let $H(x) = b(x)K(x)$, and define

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} H(y) f(x-y)\,dy = \mathbf{p}\cdot\mathbf{v}\cdot H * f(y).$$

In [5], we proved that if $\Omega \in L^q(S^{n-1})$ for some $1 < q \leq \infty$ and b is radial, then T is bounded on $L^p(R^n)$ for $1 < p < \infty$, $n \geq 2$. We recall that the space of locally integrable functions modulo constants via the norm

$$\|f\|_{\mathrm{BMO}} = \operatorname{Sup} \frac{1}{|Q|} \int_Q |f - f_Q| < \infty, \tag{0.1}$$

is called $\mathrm{BMO}(R^n)$, where the supremum is taken over all cubes Q whose sides are parallel to the Cartesian coordinates. It turns out that b must satisfy a certain degree of smoothness in order for T to be bounded from $L^{\infty}(R^n)$ to $\mathrm{BMO}(R^n)$. Let $Q_r(x)$ be the cube of side length r centered at x, whose sides are parallel to the axes, and let

$$\Phi = \left\{ \phi \in L^{\infty}(R^n):\ \text{suppt} \quad \phi \subseteq Q_1(0),\, \|\phi\|_{\infty} \leq 1,\ \int \phi = 0 \right\},$$

$\phi_r(x) = r^{-n}\phi(x/r)$. We also assume K satisfies Hörmander's condition

$$\operatorname*{Sup}_{|y|>0} \int_{|x|>2|y|} |K(x+y) - K(x)|\,dx < c. \tag{0.2}$$

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Theorem 1. *Suppose b is a bounded (not necessarily radial) and measurable function such that $H(x) = b(x)K(x)$ has a bounded Fourier transform, then the operator T maps $L^\infty(R^n)$ into* BMO(R^n) *if and only if b satisfies*

$$\text{(0.3)} \qquad \operatorname*{Sup}_{\substack{r>0 \\ \phi\in\Phi}} \|K(x)(b * \phi_r(x)\|_{L^1(R^n\setminus Q_{2\sqrt{n}r})} < \infty .$$

Since for $\Omega \in L^q(S^{n-1})$, $1 < q \le \infty$, $H(x)$ has a bounded Fourier transform for $n \ge 2$ [5], this leads to

Corollary. *Let $\Omega \in L^q(S^{n-1})$, $1 < q \le \infty$, $n \ge 2$, and $H(x) = b(|x|)K(x)$, where b is bounded. Then T maps L^∞ into* BMO *if and only if*

$$\text{(0.4)} \qquad \operatorname*{Sup}_{r>0,\phi\in\Phi} \|K(x)(b * \phi_r)(x)\|_{L^1(R^n\setminus Q_{2\sqrt{n}r})} < \infty .$$

Before we proceed with the proofs, we state the following result concerning weak type estimates; its proof is left to the reader.

We say a function $a(x)$ is a weighted 1-atom (with respect to a weight w) if

(i) suppt $a \subseteq Q$, a cube Q,
(ii) $\int a(x)\,dx = 0$,
(iii) $\int |a(x)|w(x)\,dx = 1$.

It is known that if w is an A_1 weight and Ω satisfies the Dini condition, then the operator S defined by $Sf(x) = \mathbf{p}\cdot\mathbf{v}\cdot K * f(x)$ satisfies the weighted weak type estimate

$$\text{(0.5)} \qquad w(\{x \in R^n : |Sf(x)| > \lambda\}) \le \frac{c}{\lambda}\|f\|_{L^1_w} \qquad \text{for } \lambda > 0,$$

where c is independent of $f \in L^1_w(R^n)$ [3]. It turns out that in extending the above result to our operators, b will depend on w as well as K.

Theorem 2. *Suppose $w \in A_1$, $b \in L^\infty(R^n)$, Ω satisfies the Dini condition and*

$$\text{(0.6)} \qquad \operatorname*{Sup}_Q \int_{R^n\setminus Q^*} |K(x - y_Q)(b * a)(x)|w(x)\,dx < \infty,$$

where supremum is taken over all weighted 1*-atoms (with respect to w), $Q =$* suppt *a, $y_Q =$ center of Q, and Q^* is the cube centered at y_Q with twice as large a diameter as Q. If T is bounded on $L^2_w(R^n)$, then it is of weighted weak-type* $(1, 1)$.

Let w be a weight and $L^\infty_w = \{f : \|f\|_{L^\infty_w} = \|f/w\|_\infty < \infty\}$. The weighted bounded mean oscillation space BMO_w is defined as

$$\text{BMO}_w = \{f \in L^1_{\text{loc}}(R^n) : \|f\|_{\text{BMO}_w} = \operatorname*{Sup}_Q \frac{1}{w(Q)}\int_Q |f - f_Q| < \infty\},$$

where $w(Q) = \int_Q w(x)\,dx$. Define

$$\text{(1.1)} \qquad \|f\|^*_{|BMO} = \operatorname*{Sup}_{\substack{\phi\in\Phi,\ r>0 \\ x\in R^n}} \frac{|Q_r(x)|}{w(Q_r(x))}|(f * \phi_r)(x)| .$$

If $w \in A_p$ for some $1 \le p < \infty$, then $\|f\|^*_{\mathrm{BMO}_w} \approx \|f\|_{\mathrm{BMO}_w}$, i.e.,

$$A'\|f\|_{\mathrm{BMO}_w} \le \|f\|^*_{\mathrm{BMO}_w} \le A\|f\|_{\mathrm{BMO}_w}, \tag{1.2}$$

where A and A' ae constants independent of $f \in \mathrm{BMO}_w$. We need the unweighted version of the following lemma. The more general statement is proved because it is of some interest in itself.

Lemma 1. *Let w be a weight and $K \in L^1(R^n)$ be a kernel. If $Sf = K * (f/w)$ for $f \in L^\infty_w$, then S maps L^∞_w continuously into BMO_w if and only if for each $r > 0$*

$$\operatorname*{Sup}_{\phi\in\Phi} \|K * \phi_r(x)\|_{L^1(R^n)} \le cA_r, \tag{1.3}$$

where $A_r = \mathrm{Inf}_{x\in R^n} w(Q_r(x))/|Q_r(x)|$ and the constant c is independent of $\|K\|_{L^1}$.

Proof of Lemma 1. Suppose S is bounded from L^∞_ω to BMO_w, then there is a constant c such that $\|K^*(f/w)\|_{\mathrm{BMO}_w} \le c\|f/w\|_\infty$. By (1.2) for fixed $r > 0$, $x \in R^n$,

$$\begin{aligned}\frac{|Q_r(x)|}{W(Q_r(x))}(K * (f/w) * \phi_r)(x) &\le \|K * (f/w)\|^*_{\mathrm{BMO}_w} \\ &\le A\|K * (f/w)\|_{\mathrm{BMO}_w} \\ &\le c\|f/w\|.\end{aligned}$$

Thus

$$\frac{|Q_r(x)|}{w(Q_r(x))}\int (K * \phi_r)(y)(f/w)(x-y)\,dy \le cA\|f/w\|_\infty.$$

For $x \in R^n$, choose f such that $(f/w)(x-y) = \mathrm{sgn}\ (K * \phi_r)(y)$, hence

$$\frac{|Q_r(x)|}{w(Q_r(x))}\|K * \phi_r\|_{L^1(R^n)} \le cA$$

or

$$\operatorname*{Sup}_{\phi\in\Phi} \|K * \phi_r\|_{L^1(R^n)} \le c\,\mathrm{Inf}_{x\in R^n} \frac{w(Q_r(x))}{|Q_r(x)|} = cA_r.$$

Conversely, suppose (1.3) holds. For each $\phi \in \Phi$ and each cube $Q = Q_r(x)$,

$$\frac{|Q|}{w(Q)}\int |(K * \phi_r)(y)(f/w)(x-y)|\,dy \le \frac{|Q|}{w(Q)}\|K * \phi_r\|_1\|f/w\|_\infty \le c\|f/w\|_\infty.$$

Therefore,

$$\begin{aligned}\|K * (f/w)\|_{\mathrm{BMO}_w} &\le c\|K * (f/w)\|^*_{\mathrm{BMO}} \\ &= \operatorname*{Sup}_{\substack{r>0\\ x\in R^n}} \frac{|Q|}{w(Q)}\int |(K * \phi_r)(y)(f/w)(x-y)|\,dy \\ &\le c\|f/w\|_\infty.\end{aligned}$$

Let H be a kernel such that $H^{\varepsilon}(x) = H(x)$. $\chi_{\varepsilon<|x|<1/\varepsilon}$ is integrable for each $\varepsilon > 0$. If $Sf(x) = \mathbf{p}\cdot\mathbf{v}\cdot H * f(x) = \lim_{\varepsilon\to 0} H^{\varepsilon} * f(x)$ exists, then by the dominated convergence theroem and (1.3), T is bounded from $L^{\infty}(R^n)$ to $\mathrm{BMO}(R^n)$ if and only if

$$\text{(1.4)} \qquad \operatorname*{Sup}_{\phi\in\Phi} \|S\phi_r\|_1 < \infty.$$

This lemma in the unweighted case is due to Peetre [6].

Proof of Theroem 1. Let b be a bounded function such that $H(x) = b(x)K(x)$ has a bounded Fourier transform. Set $Tf = p\cdot v\cdot H*f$. Suppose T is bounded from L^{∞} to BMO. By (1.4) $\|T\phi_r\|_1 \le c$ for all $r > 0$, $\phi\in\Phi$. In particular,

$$\text{(1.5)} \qquad \|T\phi_r\|_{L^1(R^n\setminus Q_{2\sqrt{n}r})} \le c.$$

Let $K_{\varepsilon}(x) = K(x)\cdot\chi_{|x|>\varepsilon}$. We break $T\phi_r$ into two parts

$$\begin{aligned} T\phi_r(x) &= \lim_{\varepsilon\to 0}\int b(x-y)K_{\varepsilon}(x-y)\phi_r(y)\,dy \\ &= \lim_{\varepsilon\to 0}\int b(x-y)[K_{\varepsilon}(x-y) - K_{\varepsilon}(x)]\phi_r(y)\,dy + K(x)(b*\phi_r)(x) \\ &= g(x, r) + h(x, r). \end{aligned}$$

Note that if $x\in R^n\setminus Q_{2\sqrt{n}r}$ and $y\in Q_r = Q_r(0)$, then $|x| \ge \sqrt{n}r \ge 2|y|$, hence $R^n\setminus Q_{2\sqrt{n}r} \subseteq \{x\in R^n : |x| \ge 2|y|\}$. Since K satisfies Hörmander's condition, (0.2) will be satisfied by all its truncates with a slightly larger c [8]. Now,

$$\begin{aligned} &\int_{R^n\setminus Q_{2\sqrt{n}r}} |g(x, r)|\,dx \\ &\quad = \int_{R^n\setminus Q_{2\sqrt{n}r}} \left|\lim_{\varepsilon\to 0}\int b(x-y)[K_{\varepsilon}(x-y) - K_{\varepsilon}(x)]\phi_r(y)\,dy\right| dx \\ &\quad \le \|b\|_{\infty} \operatorname*{Sup}_{\varepsilon>0}\int\left(\int_{R^n\setminus Q_{2\sqrt{n}r}} |K_{\varepsilon}(x-y) - K_{\varepsilon}(x)|\,dx\right)|\phi_r(y)|\,dy \\ &\quad \le \|b\|_{\infty} \operatorname*{Sup}_{\varepsilon>0}\int\left(\int_{|x|\ge 2|y|} |K_{\varepsilon}(x-y) - K_{\varepsilon}(x)|\,dx\right)|\phi_r(y)|\,dy \\ &\quad \le c\|b\|_{\infty}\int |\phi_r(y)|\,dy \\ &\quad = c\int |\phi(y)|\,dy = c, \end{aligned}$$

using Fubini's theorem along with the dominated convergence theorem. It follows from (1.5) that

$$\begin{aligned} \|K(x)(b*\phi_r)(x)\|_{L^1(R^n\setminus Q_{2\sqrt{n}r})} &= \|h\|_{L^1(R^n\setminus Q_{2\sqrt{n}r})} \\ &\le \|g+h\|_{L^1(R^n\setminus Q_{2\sqrt{n}r})} + \|g\|_{L^1(R^n\setminus Q_{2\sqrt{n}r})} \le c, \end{aligned}$$

hence (0.3) holds if we take supremum over all $\phi\in\Phi$ and $r > 0$. Conversely, suppose (0.3) holds. Once again $\|T\phi_r\|_{L^1(R^n\setminus Q_{2\sqrt{n}r})} = \|g+h\|_{L^1(R^n\setminus Q_{2\sqrt{n}r})} \le \|g\|_{L^1(R^n\setminus Q_{2\sqrt{n}r})} + \|h\|_{L^1(R^n\setminus Q_{2\sqrt{n}r})} \le c$. Also $\|T\phi_r\|_{L^1(Q_{2\sqrt{n}r})} = \|T\phi_r \cdot \chi_{Q_{2\sqrt{n}r}}\|_1 \le \|T\phi_r\|_{L^2(R^n)} \cdot \|\chi_{Q_{2\sqrt{n}r}}\|_{L^2(R^n)} = \|\widehat{H}\cdot\hat{\phi}_r\|_{L^2(R^n)} \cdot \mathrm{cr}^{n/2} \le \mathrm{cr}^{n/2} \sup_\xi |\widehat{H}(\xi)| \cdot \|\hat{\phi}_r\|_{L^2(R^n)} = \mathrm{cr}^{n/2}\|\phi_r\|_{L^2(R^n)} = \mathrm{cr}^{n/2}\cdot r^{-n/2} = c$, using Schwarz's inequality and Plancherel's theorem. Thus $\|T\phi_r\|_1 = \|T\phi_r\|_{L^1(R^n\setminus Q_{2\sqrt{n}r})} + \|T\phi_r\|_{L^1(Q_{2\sqrt{n}r})} \le c$. By (1.4), T is bounded. This completes the proof.

Next we show that there are bounded radial functions $b(|x|)$ for which (0.3) fails. Hence T, in general, does not map L^∞ into BMO. This is interesting in the light of the fact that if the kernel satisfies sufficiently strong regularity and cancellation properties, the L^2 bound is actually equivalent to the continuity of the mapping $L^\infty \to \mathrm{BMO}$. See e.g. the T1 theorem of David and Journé [3]. Let

$$n = 2 \qquad \text{and} \qquad K(x_1, x_2) = \frac{x_1}{(x_1+x_2)^{3/2}}$$

be the Riesz kernel. In polar coordinates it can be written as $K(r,\theta) = (\cos\theta/r^2)\cdot K$ is smooth away from zero. Define $b(|t|)$ and $\phi(x_1,x_2)$ as follows,

$$b(t) = \begin{cases} 1 & \text{if } 10m < |t| < 10m+2 \\ -1 & \text{if } 10m-2 < |t| < 10m \\ 0 & \text{otherwise} \end{cases}$$

where m runs through positive integers and

$$\phi(x_1,x_2) = \begin{cases} -1 & \text{if } 0 < x_1 < \frac{1}{2}, \quad -\frac{1}{2} < x_2 < \frac{1}{2} \\ 1 & \text{if } -\frac{1}{2} < x_1 < 0, \quad -\frac{1}{2} < x_2 < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Claim. There are constants δ, η, and $c > 0$ such that $b * \phi(x) \ge c$ for $x \in \Gamma = \{x = re^{i\theta} : |\theta| < \delta,\ |r - 10m| < \eta \text{ and } m \ge 10\}$. Assuming the claim, it follows that

$$\begin{aligned}\int_{R^2\setminus Q_{2\sqrt{2}}} |K(x)(b*\phi)(x)|\,dx &\ge \int_\Gamma |K(x)b*\phi(x)|\,dx \\ &\ge c\int_\Gamma |K(x)|\,dx \\ &\ge c\sum_{m=10}^\infty \int_{\substack{|r-10m|<\eta \\ |\theta|<\delta}} \frac{1}{|x|^2}\,dx \\ &= c\sum_{m=10}^\infty \int_{10m-\eta}^{10m+\eta} \frac{dr}{r} \\ &\ge c\sum_{m=10}^\infty \frac{1}{10m+\eta} = \infty.\end{aligned}$$

Let $\delta = 1/100$, $\eta = 1/10$. If $x^0 = (x_1^0, x_2^0) = r_0 e^{i\theta_0} \in \Gamma$, then $(b * \phi)(x^0) = \int_Q b(y)\phi(x^0 - y)\,dy$, where $Q = Q_1(x^0)$. By the particular choice of b, ϕ, δ and η it is easy to see that $\phi(x^0 - y)b(y) = 1$ on $D = \{(x_1, x_2) \in Q: |x_1 - x_1^0| > \frac{1}{5}\}$. Therefore,

$$\begin{aligned}(b^*\phi)(x^0) &\geq \int_D b(y)\phi(x^0 - y)\,dy - \left|\int_{Q\backslash D} b(y)\phi(x^0 - y)\,dy\right| \\ &\geq |D| - |Q\backslash D| \\ &= \tfrac{3}{5} - \tfrac{2}{5} = \tfrac{1}{5}.\end{aligned}$$

Thus we can take $c = \frac{1}{5}$, and the claim is proved.

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