PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 2, February 1990

ON COMMON FIXED POINTS OF LINEAR CONTRACTIONS

ZOLTÁN SASVÁRI

(Communicated by Paul S. Muhly)

ABSTRACT. In this note we give a short and direct proof of a result about convex semigroups of linear contractions.

It is the aim of the present note to give a short proof of the following:

Theorem. *Let B be a Banach space such that B and its dual B^* are strictly convex. If S is a weakly compact convex semigroup of linear contractions on B then there exists a unique projection $s_0 \in S$ such that $s_0 s = s s_0 = s_0$ holds for all $s \in S$. Moreover, $s_0 B$ is the set of common fixed points of the operators in S.*

Recall that a Banach space is called strictly convex if $\|x\| = \|y\|$ and $x \neq y$ imply $\|(x+y)/2\| < \|x\|$. The theorem above follows immediately from Corollary 4.14 and Theorem 7.2 in [1] but it appears that no direct proof of it has been published. A short proof in a special case was found by Radjavi and Rosenthal [3, Corollary 2]. The method of deLeeuw and Glicksberg is based on the theory of operator semigroups while in [3] Schauder's fixed point theorem is used. Our argument is based on the following simple fact: if K is a convex subset of B (B^*) and K is compact in the weak (weak*) topology then there exists a unique element of K with minimal norm.

For applications of the theorem we refer to [2, 4] (see also [1, §7] for connections with ergodic theory).

Proof of the theorem. Denote by B_0 the set of common fixed points of the operators in S. Plainly $0 \in B_0$ and B_0 is a closed S-invariant subspace of B. Setting $B_0^\perp := \{l \in B^* : l(x) = 0 \text{ for all } x \in B_0\}$ and $S^* := \{s^* : s \in S\}$ we observe that $B_0^\perp$ is a closed S^*-invariant subspace of B^* and that S^* is a weak* compact convex semigroup of contractions on B^*. For every $l \in B_0^\perp$ the set $S^* l$ is convex and weak* compact and hence there exists a unique $l_0 \in S^* l$ with minimal norm. In view of $s^* l_0 \in S^* l$ and $\|s^* l_0\| \leq \|l_0\|$ we must have

Received by the editors January 27, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47D05; Secondary 54H25, 47H10.

$s^* l_0 = l_0 \ (s^* \in S^*)$, that is

$$l_0(sx) = l_0(x), \qquad s \in S, \quad x \in B. \tag{1}$$

We prove that $l_0 = 0$. Let $x \in B$ be arbitrary and consider the weakly compact convex set Sx. By the same argument as above we see that there exists $x_0 \in Sx$ having minimal norm and hence satisfying $sx_0 = x_0 \ (s \in S)$. That is $x_0 \in B_0$ and therefore $l_0(x_0) = 0$. Using (1) and the fact that $x_0 = s'x$ with some $s' \in S$ we obtain $0 = l_0(x_0) = l_0(s'x) = l_0(x)$. Thus $l_0 = 0$, i.e., $0 \in S^* l$ for every $l \in B_0^{\perp}$.

Let now $l_1, \dots, l_n \in B_0^{\perp}$ be arbitrary and choose $s_1^* \in S^*$ so that $s_1^* l_1 = 0$. Next we choose $s_2^* \in S^*$ such that $s_2^*(s_1^* l_2) = 0$. Continuing this process we obtain an operator $s^* = s_n^* \cdots s_2^* s_1^* \in S^*$ with $s^* l_i = 0 \ (i = 1, \dots, n)$. A simple compactness argument shows the existence of an operator $s_0^* \in S^*$ such that $s_0^* l = 0$ for all $l \in B_0^{\perp}$, i.e. $l(s_0 x) = 0 \ (l \in B_0^{\perp}, x \in B)$. It follows that $s_0 B = B_0$. Using the relation $sx = x \ (x \in B_0, s \in S)$ we obtain $ss_0 = s_0$.

It remains to prove that $s_0 s = s_0$. The argument is suggested by that of Corollary 4.13 in [1]. Note first that $s_0 s s_0 = s_0$ and $s_0 s s_0 s = s_0 s$ hold because of $ss_0 = s_0$. For every $l \in B^*$ we have

$$\|s^* s_0^* l\| = \|s^* s_0^* s^* s_0^* l\| \le \|s_0^* s^* s_0^* l\| \le \|s^* s_0^* l\|. \tag{2}$$

But $s_0^* s^* s_0^* = s_0^*$ and therefore (2) gives $\|s^* s_0^* l\| = \|s_0^* l\|$. If $s_0^* l \ne s^* s_0^* l$ for some $l \in B^*$ then we would have

$$\|s_0^* l\| = \|s^* s_0^* l\| = \tfrac{1}{2}\|s_0^*(s^* s_0^* l + s_0^* l)\| \le \tfrac{1}{2}\|s^* s_0^* l + s_0^* l\| < \|s_0^* l\|.$$

This contradiction shows that $s_0^* = s^* s_0^*$ and therefore $s_0 s = s_0$. The uniqueness of s_0 follows at once from $ss_0 = s_0 s = s_0$. The proof is complete.

The theorem has the following immediate corollary which generalizes Theorem 1 in [3].

Corollary. *Let B and S be as in the theorem. Then the operators in S have a common fixed point other than 0 if and only if the operator 0 is not in S.*

References

1. K. deLeeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Math. **105** (1961), 63–97.

2. ——, *The decomposition of certain group representations*, J. Analyse Math. **15** (1965), 135–192.

3. H. Radjavi and P. Rosenthal, *On fixed points of linear contractions*, Proc. Amer. Math. Soc. **93** (1985), 640–642.

4. Z. Sasvári, *Decomposition of positive definite functions defined on a neighborhood of the identity*, Monatshefte Math. **104** (1987), 139–148.

Sektion Mathematik, Technische Universität, DDR-8027 Dresden, German Democratic Republic