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ALMOST EVERYWHERE CONVERGENCE OF VILENKIN-FOURIER SERIES OF H^1 FUNCTIONS

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ABSTRACT. In [5] Ladhawala and Pankratz proved that if f is in dyadic H^1, then any lacunary sequence of partial sums of the Walsh-Fourier series of f converges a.e. We generalize their theorem to Vilenkin-Fourier series. In obtaining this result, we prove a vector-valued inequality for the partial sums of Vilenkin-Fourier series.

1. INTRODUCTION

Let $G = \prod_{i=0}^{\infty} Z_{p_i}$ be the countable direct product of cyclic groups of order p_i, where $\{p_i\}_{i\geq 0}$ is a sequence of integers with $p_i \geq 2$, and μ be the Haar measure on G normalized by $\mu(G) = 1$. G can be identified with the unit interval $(0, 1)$ in the following manner. Set $m_0 = 1$, $m_k = \prod_{i=0}^{k-1} p_i$, $k = 1, 2, \ldots$. We associate with each $\{x_i\} \in G$, $0 \leq x_i < p_i$, the point $\sum_{i=0}^{\infty} x_i m_{i+1}^{-1} \in (0, 1)$. If we disregard the countable set of p_i-rationals, this mapping is one-one, onto, and measure-preserving.

For $x = \{x_i\} \in G$, let $\phi_k(x) = \exp(2\pi i x_k/p_k)$, $k = 0, 1, \ldots$. We consider all finite products $\{\chi_n\}$ of $\{\phi_k\}$, enumerated according to a scheme of Paley. We express each nonnegative integer n as a finite sum $n = \sum_{k=0}^{\infty} \alpha_k m_k$, with $0 \leq \alpha_k < p_k$, and define $\chi_n = \prod_{k=0}^{\infty} \phi_k^{\alpha_k}$. The functions $\{\chi_n\}$ are the characters of G, and they form a complete orthonormal system on G. For the case $p_i = 2$, $i = 0, 1, \ldots, G$ is the dyadic group, $\{\phi_k\}$ are the Rademacher functions, and $\{\chi_n\}$ are the Walsh functions. In general, the system $(G, \{\chi_n\})$ is a realization of the Vilenkin systems studied in [7].

We consider the Fourier series with respect to $\{\chi_n\}$. For $f \in L^1(G)$, let

$$S_n f(x) = \int_G f(t) \sum_{j=0}^{n-1} \chi_j(x - t)\, d\mu(t), \qquad n = 1, 2, \ldots,$$

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be the nth partial sum of the Vilenkin-Fourier series of f. We define H^1 in terms of the m_kth partial sums $S_{m_k}f$, which are special for the Vilenkin-Fourier series. Let $\{G_k\}$ be a sequence of subgroups of G defined by

$$G_0 = G, \qquad G_k = \prod_{i=0}^{k-1}\{0\} \times \prod_{i=k}^{\infty} Z_{p_i}, \qquad k = 1, 2, \dots,$$

and $\mathscr{F}_k$ be the σ-algebra generated by the cosets of G_k. On the interval $(0, 1)$, atoms of $\mathscr{F}_k$ are intervals of the form $(jm_k^{-1}, (j+1)m_k^{-1})$, $j = 0, 1, \dots, m_k - 1$. We note that $\{\mathscr{F}_k\}$ is an increasing sequence of σ-algebras. Since $S_{m_k}f$ is the average of f over the cosets of G_k (see, e.g. [8, p. 312]), $\{S_{m_k}f, \mathscr{F}_k\}$ is a martingale.

For $f \in L^1(G)$, let $f^* = \sup_{k\geq 0}|S_{m_k}f|$ and

$$S(f) = \left[\sum_{k=-1}^{\infty}(S_{m_{k+1}}f - S_{m_k}f)^2\right]^{1/2}, \qquad \text{where } S_{m_{-1}}f = 0.$$

Applying Davis' result for martingales [1], we know that there exist positive constants c and C (independent of the orders $\{p_i\}$) such that

$$c\|S(f)\|_1 \leq \|f^*\|_1 \leq C\|S(f)\|_1. \tag{1.1}$$

We say that $f \in H^1(G)$ if $S(f) \in L^1(G)$, or, equivalently, $f^* \in L^1(G)$, and we write

$$\|f\|_{H^1} = \|S(f)\|_1.$$

Our definition of $H^1(G)$ is a special case of the definition of H^1 for martingales given by Garsia [3]. For other definitions of $H^1(G)$, see [2] and [6].

We have the following theorem concerning the a.e. convergence of Vilenkin-Fourier series of functions in $H^1(G)$.

Theorem 1. *Let $f \in H^1(G)$ and let $\{n_k\}_{k\geq 0}$ be a sequence of positive integers such that $m_k \leq n_k < m_{k+1}$, $k = 0, 1, \dots$. Then, as $k \to \infty$, $S_{n_k}f(x) \to f(x)$ for a.e. $x \in G$.*

For the case where G is the dyadic group, the theorem is proved by Ladhawala and Pankratz [5]. If $f \in L^p(G)$, $1 < p < \infty$, then $f^* \in L^p(G)$ by Doob's inequality. Hence $L^p(G) \subset H^1(G)$. Even for the case where $f \in L^p(G)$, $1 < p < \infty$, our result is new if the orders of the cyclic groups are unbounded, i.e. $\sup_i p_i = \infty$. For the bounded case, $\sup_i p_i < \infty$, Gosselin [4] showed that if $f \in L^p(G)$, $1 < p < \infty$, the full sequence of partial sums $\{S_n f\}$ converges a.e. to f.

To prove Theorem 1, we shall show that, for any sequence $\{n_k\}$ with $m_k \leq n_k < m_{k+1}$, $k = 0, 1, \dots$, we have

$$\mu\left\{x \in G\colon \sup_{k\geq 0}|S_{n_k}f(x)| > y\right\} \leq Cy^{-1}\|f\|_{H^1}, \tag{1.2}$$

where $y > 0$, $f \in H^1(G)$, and C is an absolute constant independent of the orders $\{p_i\}$. Since $S_{m_k} f$ converges to f in the H^1 norm, Theorem 1 will then follow by the usual density argument.

We shall obtain (1.2) as a consequence of a vector-valued inequality concerning the partial sums of Vilenkin-Fourier series.

Theorem 2. *There exist constants C and C_p such that, for any sequence $\{f_l\}$ of functions in $L^1(G)$ and any sequence of positive integers $\{n_l\}$,*

(1.3)
$$\mu\left\{x \in G\colon \left(\sum_{l=0}^{\infty} |S_{n_l} f_l(x)|^2\right)^{1/2} > y\right\} \le C y^{-1} \left\| \left(\sum_{l=0}^{\infty} |f_l|^2\right)^{1/2} \right\|_1, \qquad y > 0,$$

$$(1.4) \qquad \left\| \left(\sum_{l=0}^{\infty} |S_{n_l} f_l|^2\right)^{1/2} \right\|_p \le C_p \left\| \left(\sum_{l=0}^{\infty} |f_l|^2\right)^{1/2} \right\|_p, \qquad 1 < p < \infty.$$

The constants C and C_p are independent of the orders $\{p_i\}$ of the cyclic groups.

In what follows, C will denote an absolute constant, which may vary from line to line.

2. A decomposition lemma

To prove Theorem 2, we use a Calderón-Zygmund type decomposition lemma. This lemma is a modified version of the one given in [8; Lemma 2, p. 314]. We shall describe it on the interval $(0, 1)$.

Lemma 3. *Let $y > 0$ and $\{f_l\}_{l\ge 0}$ be a sequence of functions on G such that $\|(\sum_l |f_l|^2)^{1/2}\|_1 \le y$. Let $\{\alpha_{lk}\}_{l,k\ge 0}$ be a double sequence of integers with $0 \le \alpha_{lk} < p_k$. Then there are sequences of functions $\{g_l\}_{l\ge 0}$, $\{b_l\}_{l\ge 0}$ on G and a collection $\mathscr{C} = \{\omega_j\}$ of disjoint intervals such that*

$$(2.1) \qquad f_l = g_l + b_l, \qquad l = 0, 1, \ldots.$$

$$(2.2) \qquad \left(\sum_l |g_l|^2\right)^{1/2} \le C y \quad a.e.$$

$$(2.3) \qquad \left\| \left(\sum_l |g_l|^2\right)^{1/2} \right\|_1 \le C \left\| \left(\sum_l |f_l|^2\right)^{1/2} \right\|_1 .$$

$$(2.4) \qquad \mathscr{C} = \bigcup_{k=0}^{\infty} \mathscr{C}_k,$$

where each $\omega_j \in \mathscr{C}_k$ is measurable with respect to $\mathscr{F}_{k+1}$, and is a proper subset of a coset of G_k.

$$(2.5) \qquad b_l(x) = 0 \quad if \quad x \notin \Omega \equiv \bigcup_j \omega_j, \qquad l = 0, 1, \ldots.$$

For each $l = 0, 1, \ldots,$

$$\int_{\omega_j} b_l \, d\mu = 0 \quad \textit{for every } \omega_j \in \mathscr{C}, \textit{ and} \tag{2.6}$$

$$\int_{\omega_j} b_l \phi_k^{\alpha_{lk}} \, d\mu = 0 \quad \textit{for every } \omega_j \in \mathscr{C}_k, \; k = 0, 1, \ldots.$$

$$\int_{\omega_j} \left(\sum_l |b_l|^2 \right)^{1/2} d\mu \le C \int_{\omega_j} \left(\sum_l |f_l|^2 \right)^{1/2} d\mu \quad \textit{for every } \omega_j \in \mathscr{C}. \tag{2.7}$$

$$\sum_j \mu(\omega_j) \le y^{-1} \left\| \left(\sum_l |f_l|^2 \right)^{1/2} \right\|_1. \tag{2.8}$$

Proof. We apply the decomposition in [8] (see proof of Lemma 2, pp. 314–315) to the function $(\sum_l |f_l|^2)^{1/2}$ to obtain a collection $\mathscr{C} = \{\omega_j\}$ of disjoint intervals with the properties that

$$y < \frac{1}{\mu(\omega_j)} \int_{\omega_j} \left(\sum_l |f_l|^2 \right)^{1/2} d\mu \le 3y, \qquad \omega_j \in \mathscr{C}, \tag{2.9}$$

$$\left(\sum_l |f_l(x)|^2 \right)^{1/2} \le y, \quad \text{for a.e. } x \notin \Omega, \tag{2.10}$$

and that $\mathscr{C} = \bigcup_{k=0}^{\infty} \mathscr{C}_k$, where $\{\mathscr{C}_k\}$ satisfies (2.4). The first inequality of (2.9) then implies (2.8).

Next, we decompose $f_l = g_l + b_l$, $l = 0, 1, \ldots$, with

$$g_l(x) = \begin{cases} f_l(x) & \text{if } x \notin \Omega, \\ a_{lkj} + b_{lkj} \phi_k^{-\alpha_{lk}}(x) & \text{if } x \in \omega_j \in \mathscr{C}_k, \end{cases}$$

where a_{lkj}, b_{lkj} are constants chosen in such a way that

$$\int_{\omega_j} f_l \, d\mu = \int_{\omega_j} (a_{lkj} + b_{lkj} \phi_k^{-\alpha_{lk}}) \, d\mu,$$

and

$$\int_{\omega_j} f_l \phi_k^{\alpha_{lk}} \, d\mu = \int_{\omega_j} (a_{lkj} + b_{lkj} \phi_k^{-\alpha_{lk}}) \phi_k^{\alpha_{lk}} \, d\mu.$$

Then $b_l = g_l - f_l$ satisfies (2.5) and (2.6). Also, it follows from the proof of (25) in [8, pp. 315–317] that

$$|g_l(x)| \le \frac{C}{\mu(\omega_j)} \int_{\omega_j} |f_l| \, d\mu, \qquad x \in \omega_j, \; \omega_j \in \mathscr{C}.$$

Hence, by Minkowski's inequality for integrals, we have

$$\left(\sum_l |g_l(x)|^2 \right)^{1/2} \le \frac{C}{\mu(\omega_j)} \int_{\omega_j} \left(\sum_l |f_l|^2 \right)^{1/2} d\mu, \qquad x \in \omega_j, \; \omega_j \in \mathscr{C}.$$

This inequality, together with (2.9) and (2.10), implies (2.2), (2.3) and (2.7). □

3. Proof of Theorem 2

We shall prove (1.3). The case $p = 2$ of (1.4) is a consequence of Plancherel's formula. For $1 < p < 2$, (1.4) follows from (1.3) and the case $p = 2$ by the vector-valued Marcinkiewicz interpolation theorem. The case $2 < p < \infty$ then follows by a duality argument.

Instead of proving (1.3), we shall prove an equivalent inequality involving the modified partial sums $\{S_n^* f\}$. For $f \in L^1(G)$, let

$$S_n^* f = \bar{\chi}_n S_n(f\chi_n), \qquad n = 1, 2, \ldots .$$

(For the properties of $S_n^* f$, see [8, pp. 313–314].) (1.3) is equivalent to

$$\mu\left\{x \in G\colon \left(\sum_{l=0}^{\infty} |S_{n_l}^* f_l(x)|^2\right)^{1/2} > y\right\} \le Cy^{-1}\left\|\left(\sum_{l=0}^{\infty} |f_l|^2\right)^{1/2}\right\|_1, \tag{3.1}$$

where $y > 0$, $\{f_l\}$ is any sequence of functions in $L^1(G)$, and $\{n_l\}$ is any sequence of positive integers.

To prove (3.1), we can assume $\|(\sum_l |f_l|^2)^{1/2}\|_1 \le y$. For each $l = 0, 1, \ldots,$ decompose f_l as in Lemma 3. Since

$$\begin{aligned} &\mu\left\{\left(\sum_l |S_{n_l}^* f_l|^2\right)^{1/2} > y\right\} \\ &\quad \le \mu\left\{\left(\sum_l |S_{n_l}^* g_l|^2\right)^{1/2} > y/2\right\} + \mu\left\{\left(\sum_l |S_{n_l}^* b_l|^2\right)^{1/2} > y/2\right\}, \end{aligned} \tag{3.2}$$

(3.1) will be proved if we can show that each term on the right is bounded by $Cy^{-1}\|(\sum_l |f_l|^2)^{1/2}\|_1$.

Using Plancherel's formula, we obtain

$$\begin{aligned} \mu\left\{\left(\sum_l |S_{n_l}^* g_l|^2\right)^{1/2} > y/2\right\} &\le Cy^{-2}\left\|\left(\sum_l |S_{n_l}^* g_l|^2\right)^{1/2}\right\|_2^2 \\ &\le Cy^{-2}\left\|\left(\sum_l |g_l|^2\right)^{1/2}\right\|_2^2 \\ &\le Cy^{-1}\left\|\left(\sum_l |f_l|^2\right)^{1/2}\right\|_1, \end{aligned}$$

by (2.2) and (2.3).

To estimate the second term in (3.2), we use the following notation. Let $\omega_j \in \mathscr{F}_{k+1}$, with ω_j contained in the coset I of G_k. We consider I as a circle and let ω_j^* denote the interval inside I which contains ω_j at its center with $\mu(\omega_j^*) = 3\mu(\omega_j)$. Let $\Omega^* = \bigcup_j \omega_j^*$. Then, by (2.8),

$$\mu(\Omega^*) \le 3\sum_j \mu(\omega_j) \le 3y^{-1}\left\|\left(\sum_l |f_l|^2\right)^{1/2}\right\|_1 .$$

Therefore, it suffices to prove

$$\mu\left\{x \notin \Omega^* : \left(\sum_l |S_{n_l}^* b_l(x)|^2\right)^{1/2} > y/2\right\} \le Cy^{-1}\left\|\left(\sum_l |f_l|^2\right)^{1/2}\right\|_1 . \tag{3.3}$$

To do this, we express $S_{n_l}^* b_l$ in terms of the conjugate functions. Let $f \in L^1(G)$. For $x \in \{x_k\} \in G$, $I = x + G_k$, define

$$H_k f(x) = \frac{1}{2}\frac{1}{\mu(I)}\int_{I\cap\{x_k \ne t_k\}} f(t)\cot(\pi(x_k - t_k)/p_k)\,d\mu(t).$$

If $n_l = \sum_{k=0}^{\infty} \alpha_{lk} m_k$, $0 \le \alpha_{lk} < p_k$, it is shown in [8, pp. 313–314] that

$$\begin{aligned} S_{n_l}^* b_l(x) = &\sum_{k=0}^{\infty} \frac{\alpha_{lk}}{\mu(I)}\int_{I\cap\{x_k = t_k\}} b_l(t)\,d\mu(t) \\ &+ \frac{1}{2}\sum_{k=0}^{\infty} \phi_k^{-\alpha_{lk}}(x)\frac{1}{\mu(I)}\int_{I\cap\{x_k\ne t_k\}} b_l(t)\phi_k^{\alpha_{lk}}(t)\,d\mu(t) \\ &- \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{\mu(I)}\int_{I\cap\{x_k\ne t_k\}} b_l(t)\,d\mu(t) \\ &+ i\sum_{k=0}^{\infty} \phi_k^{-\alpha_{lk}}(x) H_k(b_l\phi_k^{\alpha_{lk}})(x) \\ &- i\sum_{k=0}^{\infty} H_k b_l(x). \end{aligned}$$

For $x \notin \Omega^*$, (2.5) and (2.6) imply that the first three terms on the right vanish. (See the explanation below (29) on p. 317 in [8].) Thus, it follows from Minkowski's inequality that, for $x \notin \Omega^*$,

$$\begin{aligned} \left[\sum_l |S_{n_l}^* b_l(x)|^2\right]^{1/2} \le &\sum_{k=0}^{\infty}\left[\sum_l |H_k(b_l\phi_k^{\alpha_{lk}})(x)|^2\right]^{1/2} \\ &+ \sum_{k=0}^{\infty}\left[\sum_l |H_k b_l(x)|^2\right]^{1/2} . \end{aligned}$$

(3.3) will be proved if we show

$$\begin{aligned} &\mu\left\{x \notin \Omega^* : \sum_{k=0}^{\infty}\left[\sum_l |H_k(b_l\phi_k^{\alpha_{lk}})(x)|^2\right]^{1/2} > y/4\right\} \\ &\qquad \le Cy^{-1}\left\|\left(\sum_l |f_l|^2\right)^{1/2}\right\|_1 \end{aligned} \tag{3.4}$$

and

$$\mu\left\{x \notin \Omega^*: \sum_{k=0}^{\infty}\left[\sum_l |H_k b_l(x)|^2\right]^{1/2} > y/4\right\} \tag{3.5}$$
$$\leq Cy^{-1}\left\|\left(\sum_l |f_l|^2\right)^{1/2}\right\|_1 .$$

We shall demonstrate (3.4). (3.5) can be proved similarly.

Suppose $x \notin \Omega^*$ and $I = x + G_k$. Again it follows from (2.5) and (2.6) that

$$H_k(b_l\phi_k^{\alpha_{lk}})(x) = \frac{1}{2}\frac{1}{\mu(I)}\sum_{\omega_j \subset I;\omega_j \in \mathscr{C}_k}\int_{\omega_j} b_l(t)\phi_k^{\alpha_{lk}}(t)$$
$$\times\left[\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right)\right] d\mu(t),$$

where $t^j = \{t_k^j\}_{k\geq 0}$ is any fixed point in ω_j. (See the proof below (32) on p. 138 in [8].) By Minkowski's inequality for integrals, we have

$$\left[\sum_l |H_k(b_l\phi_k^{\alpha_{lk}})(x)|^2\right]^{1/2} \leq \frac{1}{2}\frac{1}{\mu(I)}\sum_{\omega_j \subset I;\omega_j \in \mathscr{C}_k}\int_{\omega_j}\left[\sum_l |b_l(t)|^2\right]^{1/2}$$
$$\times\left|\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right)\right| d\mu(t).$$

Thus, for any coset I of G_k, Fubini's theorem gives

$$\int_{I\cap {}^c\Omega^*}\left[\sum_l |H_k(b_l\phi_k^{\alpha_{lk}})(x)|^2\right]^{1/2} d\mu(x)$$
$$\leq \frac{1}{2}\frac{1}{\mu(I)}\sum_{\omega_j \subset I;\omega_j \in \mathscr{C}_k}\int_{\omega_j}\left[\sum_l |b_l(t)|^2\right]^{1/2}$$
$$\times\int_{I\cap {}^c\omega_j^*}\left|\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right)\right| d\mu(x)\, d\mu(t).$$

Since, for $t \in \omega_j$, a simple computation gives

$$\frac{1}{\mu(I)}\int_{I\cap {}^c\omega_j^*}\left|\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right)\right| d\mu(x) \leq C,$$

we have

$$\int_{I\cap {}^c\Omega^*}\left[\sum_l |H_k(b_l\phi_k^{\alpha_{lk}})|^2\right]^{1/2} d\mu \leq C\sum_{\omega_j \subset I;\omega_j \in \mathscr{C}_k}\int_{\omega_j}\left(\sum_l |b_l|^2\right)^{1/2} d\mu$$
$$\leq C\sum_{\omega_j \subset I;\omega_j \in \mathscr{C}_k}\int_{\omega_j}\left(\sum_l |f_l|^2\right)^{1/2} d\mu,$$

by (2.7). Therefore

$$\mu\left\{x \notin \Omega^*: \sum_{k=0}^{\infty}\left[\sum_l |H_k(b_l\phi_k^{\alpha_{lk}})(x)|^2\right]^{1/2} > y/4\right\}$$
$$\leq Cy^{-1}\sum_{k=0}^{\infty}\int_{{}^c\Omega^*}\left[\sum_l |H_k(b_l\phi_k^{\alpha_{lk}})|^2\right]^{1/2} d\mu$$
$$\leq Cy^{-1}\sum_{k=0}^{\infty}\sum_{\omega_j\in\mathscr{C}_k}\int_{\omega_j}\left(\sum_l |f_l|^2\right)^{1/2} d\mu$$
$$\leq Cy^{-1}\left\|\left(\sum_l |f_l|^2\right)^{1/2}\right\|_1 .$$

This establishes (3.4), and hence completes the proof of Theorem 2.

4. Proof of Theorem 1

It suffices to prove (1.2). Let $f \in H^1(G)$ and $f_k = S_{m_{k+1}}f - S_{m_k}f$, $k = 0, 1, \ldots$. For $m_k \leq n_k < m_{k+1}$, $S_{n_k}f = S_{m_k}f + S_{n_k}f_k$. Hence, for $y > 0$,

$$\mu\left\{\sup_{k\geq 0}|S_{n_k}f| > y\right\} \leq \mu\left\{\sup_{k\geq 0}|S_{m_k}f| > y/2\right\} + \mu\left\{\sup_{k\geq 0}|S_{n_k}f_k| > y/2\right\}.$$

By (1.1),

$$\mu\left\{\sup_{k\geq 0}|S_{m_k}f| > y/2\right\} \leq 2y^{-1}\|f^*\|_1 \leq Cy^{-1}\|f\|_{H^1}.$$

From (1.3) of Theorem 2,

$$\mu\left\{\sup_{k\geq 0}|S_{n_k}f_k| > y/2\right\} \leq \mu\left\{\left(\sum_{k=0}^{\infty}|S_{n_k}f_k|^2\right)^{1/2} > y/2\right\}$$
$$\leq Cy^{-1}\left\|\left(\sum_{k=0}^{\infty}|f_k|^2\right)^{1/2}\right\|_1 \leq Cy^{-1}\|f\|_{H^1}.$$

This completes the proof of Theorem 1.

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