PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 2, February 1990

REAL ANALYTIC BOUNDARY REGULARITY OF THE CAUCHY KERNEL ON CONVEX DOMAINS

SO-CHIN CHEN

(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. It is well-known that in one complex variable the Cauchy integral preserves real analyticity near the boundary. In this paper we show that the same conclusion also holds on convex domains with real analytic boundary in higher dimension, where the Cauchy kernel is given by the Cauchy-Fantappiè form of order zero generated by the (1.0)-form $C(\xi, z)$,

$$C(\xi, z) = \left(\sum_{j=1}^{n} \frac{\partial r}{\partial \xi_j}(\xi)\, d\xi_j\right)\left(\sum_{j=1}^{n} \frac{\partial r}{\partial \xi_j}(\xi)(\xi_j - z_j)\right)^{-1},$$

where $r(\xi)$ is the defining function of the domain.

1. INTRODUCTION

In one complex variable there is a classical theorem concerning the real analytic boundary regularity of the Cauchy kernel which states:

Theorem. *Let D be a smooth bounded domain in $\mathbf{C}$ with real analytic boundary, and let $z_0 \in bD$ be a boundary point. Let U be an open neighborhood of z_0 in $\mathbf{C}$. Suppose that $f(\zeta) \in L^2(bD) \cap C^w(bD \cap U)$. Then the integral*

$$F(z) = \int_{bD} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \text{ for } z \in D,$$

defines a holomorphic function on D that is real analytic up to the boundary on $bD \cap U$.

It would be very interesting to generalize this type of theorem to higher dimension. Unfortunately, this cannot be done in general. We do not even know how to write down an appropriate analog of the Cauchy kernel on a smooth bounded domain of holomorphy with real analytic boundary. So in this paper we are going to study this problem on convex domains, where there is a natural

Received by the editors March 15, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32A25, 32A40.
Key words and phrases. convex domain, Cauchy-Fantappiè form.

Cauchy kernel as follows: Let $C(\xi, z)$ be the generating $(1,0)$-form given by

$$C(\xi, z) = \frac{\partial r}{\phi(\xi, z)} = \frac{\sum_{j=1}^{n} \frac{\partial r}{\partial \xi_j}(\xi)\, d\xi_j}{\sum_{j=1}^{n} \frac{\partial r}{\partial \xi_j}(\xi)(\xi_j - z_j)},$$

the Cauchy kernel $\Omega_0(C)$ is then defined to be the Cauchy-Fantappiè form of order zero,

$$\Omega_0(C(\xi, z)) = C(\xi, z) \wedge (\bar{\partial}_\xi C(\xi, z))^{n-1}.$$

For more details see Range [2].

Here are our main results.

Theorem 1. *Let $D \subseteq \mathbf{C}^n$, $n \geq 2$, be a smooth bounded convex domain with real analytic defining function $r(z)$, and let $z_0 \in bD$ be a boundary point. Let U be a small open neighborhood of z_0 in $\mathbf{C}^n$. Suppose that $f(\xi) \in L^2(bD) \cap C^w(bD \cap U)$. Then the integral*

$$F(z) = \int_{bD} f(\xi)\Omega_0(C(\xi, z)), \qquad \text{for } z \in D,$$

defines a holomorphic function on D that is real analytic up to the boundary $bD \cap U$. In particular $F(z)$ can be extended holomorphically across $bD \cap U$.

Corollary 2. *Let D be defined as in Theorem 1. Suppose that $f(\xi) \in C^w(bD)$. Then $F(z) \in C^w(\overline{D})$. In particular $F(z) \in H(\overline{D})$.*

We make a remark that the proof we present here for Theorem 1 also works for the case $n = 1$, and if $n = 1$ the Cauchy kernel $\Omega_0(C)$ reduces to the classical one,

$$\Omega_0(C) = \frac{\frac{\partial r}{\partial \xi}(\xi)\, d\xi}{\frac{\partial r}{\partial \xi}(\xi)(\xi - z)} = \frac{d\xi}{\xi - z}.$$

II. Proof of Theorem 1

Let $D \subseteq \mathbf{C}^n$, $n \geq 2$, be a smooth bounded convex domain with real analytic defining function $r(z)$. The Cauchy kernel on D is defined as follows. For $\xi \in bD$ and $z \in D$ one defines the generating $(1,0)$-form $C(\xi, z)$ by

$$C(\xi, z) = \frac{\partial r}{\phi(\xi, z)},$$

where $\partial r(\xi) = \sum_{j=1}^{n} \frac{\partial r}{\partial \xi_j}(\xi)\, d\xi_j$ and $\phi(\xi, z) = \sum_{j=1}^{n} \frac{\partial r}{\partial \xi_j}(\xi)(\xi_j - z_j)$.

The Cauchy kernel $\Omega_0(C)$ for D is then defined to be the Cauchy-Fantappiè form of order zero as follows:

$$\begin{aligned}\Omega_0(C(\xi,z)) &= C(\xi,z)\wedge(\bar\partial_\xi C(\xi,z))^{n-1}\\ &= \frac{(\partial r)\wedge(\bar\partial\partial r)^{n-1}}{\phi^n(\xi,z)}\\ &= \frac{\left(\sum_{j=1}^n \frac{\partial r}{\partial\xi_j}d\xi_j\right)\wedge\left(\sum_{j,k=1}^n \frac{\partial^2 r}{\partial\bar\xi_k\,\partial\xi_j}d\bar\xi_k\,d\xi_j\right)^{n-1}}{\phi^n(\xi,z)}\\ &= \sum_{\substack{\text{finite sum}\\ |I|=n-1}} \frac{A_I(\xi)}{\phi^n(\xi,z)}d\xi_1\dots d\xi_n\,d\bar\xi_I,\end{aligned}$$

where I denotes the increasing multiindices of order $n-1$, and $A_I(\xi)$ is a polynomial that involves the first and second derivatives of the defining function $r(\xi)$. Hence $A_I(\xi)$ is real analytic in some open neighborhood of $\overline{D}$.

The convexity of the domain D implies $\phi(\xi,z)\neq 0$ for $\xi\in bD$ and $z\in D$. Therefore it makes sense to define the following integral for $f(\xi)\in L^2(bD)$,

$$F(z)=\int_{bD} f(\xi)\Omega_0(C(\xi,z)). \tag{2.1}$$

Since $\phi(\xi,z)$ is holomorphic in z, we see that $F(z)\in H(D)$.

The main purpose of this paper is to show real analytic boundary regularity of this integral. Namely, if $z_0\in bD$ is a fixed boundary point and $f(\xi)$ is also real analytic in some open neighborhood of z_0 in the boundary, then $F(z)$ is also real analytic up to the boundary near z_0. In particular it follows that $F(z)$ can be extended holomorphically across the boundary near z_0.

Before we begin to prove Theorem 1, we will state several lemmas concerning the function $\phi(\xi,z)$ that are needed in the sequel. First the following lemma is proved in Range [1].

Lemma 2.2. *Let D and $\phi(\xi,z)$ be defined as above, and let $z_0\in bD$. Then there exists a positive integer m_0 and constants A, δ_1, $\delta_2>0$ such that*

$$|\phi(\xi,z)|\geq A(d(z,bD)+|\operatorname{Im}\phi(\xi,z)|+|\xi-z|^{m_0}), \tag{2.3}$$

for $\xi\in bD\cap B(z_0,\delta_1)$ and $z\in\overline{D}\cap B(\xi,\delta_2)$, where $d(z,bD)$ denotes the distance between z and the boundary bD.

Lemma 2.4. *Let D and $\phi(\xi,z)$ be defined as above, and let $z_0\in bD$. For $\delta\ll\min(\delta_1,\delta_2)$, we have for some constant $C>0$,*

$$\int_{bD\cap B(z_0,\delta)}|\log\phi(\xi,z)|\,dS\leq C$$

uniformly for all $z\in D\cap B(z_0,\delta)$.

Proof. First by a standard choice (e.g. see Range [2]) one may introduce a local coordinate system $(t_1, \dots, t_{2n}) = t = t(\xi, z)$ on $B(z, \eta)$, for some $\eta > 0$, such that the following hold.

(i) $t_1(\xi, z) = r(\xi)$ and $t(z, z) = (r(z), 0, \dots, 0)$,
(ii) $t_2(\xi, z) = \operatorname{Im}\phi(\xi, z)$,
(iii) $|t(\xi, z)| < 1$ for $\xi \in B(z, \eta)$,
(iv) The Jacobian is bounded both from above and from below uniformly for all $z \in \overline{D} \cap B(z_0, \delta)$.

Hence by Lemma 2.2 for sufficiently small δ we have

$$\begin{aligned} |\log \phi(\xi, z)| &\lesssim |\log|\phi(\xi, z)|| + |\arg \phi(\xi, z)| \\ &\lesssim |\log A| + |\log|\xi - z|^{m_0}| + |\arg\phi(\xi, z)|. \end{aligned}$$

Since we have $\frac{1}{2}\pi \le \arg\phi(\xi, z) \le \frac{3}{2}\pi$ for all $z \in D \cap B(z_0, \delta)$ and $\xi \in bD \cap B(z_0, \delta)$, hence by setting $t' = (t_2, \dots, t_{2n})$ we obtain

$$\begin{aligned} \int_{bD\cap B(z_0, \delta)} |\log\phi(\xi, z)|\, dS &\lesssim 1 + \int_{bD\cap B(z_0, \delta)} |\log|\xi - z|^{m_0}|\, dS \\ &\lesssim 1 + \int_{|t'|<1} |\log|t'|^{m_0}|\, dt_2 \dots dt_{2n} \\ &\lesssim 1 + \int_0^1 |\log(r^{m_0})| \cdot r^{2n-2}\, dr \\ &\lesssim 1. \end{aligned}$$

The last inequality holds because $n \ge 2$. This also completes the proof of the lemma.

Now we are ready to prove our main result. In fact what we are going to prove is a more general setting, namely,

Theorem 2.5. *Let D and $\phi(\xi, z)$ be defined as above, and let $z_0 \in bD$. If $A(\xi) \in C^w(bD)$ and $f(\xi) \in L^2(bD) \cap C^w(bD \cap U)$, where U is an open neighborhood of z_0 in $\mathbf{C}^n$, then for each increasing multiindex I with $|I| = n - 1$ and each integer $m \in \mathbf{N}$, the integral*

$$F(z) = \int_{bD} f(\xi) \frac{A(\xi)}{\phi^m(\xi, z)}\, d\xi_1 \dots d\xi_n\, d\bar{\xi}_I \tag{2.6}$$

defines a holomorphic function on D that is real analytic up to the boundary on $bD \cap U$.

Theorem 1 will then follow immediately from Theorem 2.5.

Proof of Theorem 2.5. It is clear that $F(z)$ is holomorphic on D. Hence the conclusion will follow by Sobolev's embedding theorem if one can show that the following estimate

$$\left|\frac{\partial^\alpha}{\partial z^\alpha} F(z)\right| \le C C^{|\alpha|} \cdot |\alpha|!, \tag{2.7}$$

holds uniformly for all $z \in D \cap$ (some neighborhood of z_0 in $\mathbf{C}^n$), and for all multiindices α.

To prove (2.7) we first fix two open balls $B_j = B(z_0, \varepsilon_j)$, $j = 1, 2$, with $\varepsilon_1 < \varepsilon_2$ and such that $\overline{B}_2 \subseteq U$. Put $V_j = bD \cap B_j$, $j = 1, 2$. Then by a lemma due to Ehrenpreis there exists a constant $M_0 > 0$ such that for every positive integer k one can find $\phi_k \in C_0^\infty(V_2)$ with $0 \le \phi_k \le 1$, $\phi_k \equiv 1$ on V_1 and

$$|D^\alpha \phi_k| \le M_0 (M_0 k)^{|\alpha|}, \text{ for } |\alpha| \le k. \tag{2.8}$$

For an outline of the proof of this lemma see Tartakoff [3].

So one can decompose $F(z)$ into two parts, i.e., $F(z) = F_1(z) + F_2(z)$, where

$$F_1(z) = \int_{bD} \phi_k(\xi) f(\xi) \frac{A(\xi)}{\phi^m(\xi, z)} d\xi_1 \ldots d\xi_n d\bar{\xi}_I,$$

and

$$F_2(z) = \int_{bD} (1 - \phi_k(\xi)) f(\xi) \frac{A(\xi)}{\phi^m(\xi, z)} d\xi_1 \ldots d\xi_n d\bar{\xi}_I.$$

It follows from the estimate (2.3) and the convexity of the domain D that there exists a constant $C_1 > 0$ such that

$$|D_z^\alpha F_2(z)| \le C_1 C_1^{|\alpha|} \cdot |\alpha|!$$

uniformly for all $z \in D \cap B(z_0, \varepsilon)$ with $0 < \varepsilon \ll \varepsilon_1$ and all k.

The remainder of this paper is devoted to the estimate of $F_1(z)$. First we show that—perhaps with a larger constant M—the function $\phi_k(\xi) f(\xi) A(\xi)$ also satisfies the estimate (2.8).

Lemma 2.9. *There exists a constant $M > 0$ such that for each multiindex α with $|\alpha| \le k$, we have*

$$|D^\alpha (\phi_k(\xi) f(\xi) A(\xi))| \le M (Mk)^{|\alpha|}. \tag{2.10}$$

Proof of Lemma 2.9. Since $f(\xi) A(\xi)$ is real analytic on $bD \cap U$, there exists a $R > 0$ such that for all $\xi \in V_2$ we have

$$|D^\beta (f(\xi) A(\xi))| \le R R^{|\beta|} \cdot |\beta|!$$

for all multiindices β. So

$$\begin{aligned}|D^{\alpha}(\phi_k(\xi)f(\xi)A(\xi))| &\le \sum_{\substack{j=0\\ \alpha_1+\alpha_2=\alpha\\ |\alpha_1|=j\\ |\alpha_2|=|\alpha|-j}}^{|\alpha|} \binom{|\alpha|}{j} |D^{\alpha_1}\phi_k||D^{\alpha_2}(f(\xi)A(\xi))| \\ &\le \sum_{j=0}^{|\alpha|} \frac{|\alpha|!}{j!(|\alpha|-j)!}\cdot M_0(M_0k)^j\cdot RR^{|\alpha|-j}\cdot(|\alpha|-j)! \\ &= M_0(M_0k)^{|\alpha|}\cdot\sum_{j=0}^{|\alpha|}\frac{|\alpha|\cdots(j+1)}{k^{|\alpha|-j}}\cdot R\cdot\left(\frac{R}{M_0}\right)^{|\alpha|-j} \\ &\le M_0(M_0k)^{|\alpha|}\cdot\sum_{l=0}^{\infty}R\cdot\left(\frac{1}{2}\right)^l \\ &\le (2RM_0)(M_0k)^{|\alpha|},\end{aligned}$$

if we choose $M_0 \ge 2R$. So if we let our new M be $\max(2RM_0, M_0)$, we are done. This completes the proof of Lemma 2.9.

Therefore from now on we will call $\phi_k(\xi)f(\xi)A(\xi)$ by $g_k(\xi)$ such that the estimate (2.10) holds for $g_k(\xi)$. Next we consider the operation of $\frac{\partial}{\partial z_j}$ on $F_1(z)$ for all $z \in D\cap B(z_0, \varepsilon)$.

$$(2.11)\qquad \begin{aligned}\frac{\partial}{\partial z_j}F_1(z) &= \int_{bD}\frac{\partial}{\partial z_j}\left(g_k(\xi)\cdot\frac{1}{\phi^m(\xi,z)}\right)d\xi_1\cdots d\xi_n d\bar{\xi}_I \\ &= \int_{bD} g_k(\xi)\left(\frac{m\dfrac{\partial r}{\partial \xi_j}(\xi)}{\phi^{m+1}(\xi,z)}\right)d\xi_1\cdots d\xi_n d\bar{\xi}_I.\end{aligned}$$

Put $T = \sum_{j=1}^n \frac{\partial r}{\partial \bar{\xi}_j}\frac{\partial}{\partial \xi_j} - \sum_{j=1}^n \frac{\partial r}{\partial \xi_j}\frac{\partial}{\partial \bar{\xi}_j}$. We see that T is tangent to the boundary and satisfies

$$T\phi(\xi, z) = \sum_{j=1}^n\left(T\left(\frac{\partial r}{\partial \xi_j}\right)\right)(\xi_j - z_j) + \sum_{j=1}^n\left|\frac{\partial r}{\partial \xi_j}\right|^2.$$

It shows that $T\phi(\xi, z)$ does not vanish near the diagonal of $bD\times bD$. So we may assume that $T\phi(\xi, z)\ne 0$ on $U_z\times U_\xi$, where U is given in Theorem 1 or Theorem 2.5. Let the adjoint T^* of T be given by

$$T^* = -T + h(\xi),$$

where $h(\xi)$ is an analytic function defined on U.

Then by integration by parts (2.11) becomes

$$
\begin{aligned}
\frac{\partial}{\partial z_j}F_1(z) &= \int_{bD} g_k(\xi)\left(-\frac{\frac{\partial r}{\partial \xi_j}}{T\phi}T\left(\frac{1}{\phi^m(\xi,z)}\right)\right)d\xi_1\cdots d\xi_n\,d\bar{\xi}_I \\
&= \int_{bD}\left(T^*\left(-\frac{\frac{\partial r}{\partial \xi_j}}{T\phi}\cdot g_k(\xi)\right)\right)\frac{1}{\phi^m(\xi,z)}d\xi_1\cdots d\xi_n\,d\bar{\xi}_I \qquad (2.12)\\
&= \int_{bD}((a_j(\xi,z)T+b_j(\xi,z))g_k(\xi))\frac{1}{\phi^m(\xi,z)}d\xi_1\cdots d\xi_n\,d\bar{\xi}_I,
\end{aligned}
$$

where $a_j(\xi,z)=(\frac{\partial r}{\partial \xi_j})(T\phi)^{-1}$ and $b_j(\xi,z)=T(\frac{\partial r}{\partial \xi_j}(T\phi)^{-1})-h\cdot(\frac{\partial r}{\partial \xi_j})(T\phi)^{-1}$. We see that both $a_j(\xi,z)$ and $b_j(\xi,z)$ are real analytic functions in ξ and z on $U_z\times U_\xi$ for $j=1,\dots,n$. Since both $a_j(\xi,z)$ and $b_j(\xi,z)$ involve variables ξ and z, then if we apply another derivative $\frac{\partial}{\partial z_i}$, $i=1,\dots,n$, we will end up with two terms as follows:

(2.13)

$$
\begin{aligned}
\frac{\partial^2}{\partial z_i\partial z_j}F_1(z) &= \frac{\partial}{\partial z_i}\int_{bD}((a_j(\xi,z)T+b_j(\xi,z))g_k(\xi))\frac{1}{\phi^m(\xi,z)}d\xi_1\dots d\xi_n\,d\bar{\xi}_I \\
&= \int_{bD}\left(\frac{\partial}{\partial z_i}(a_j(\xi,z)T+b_j(\xi,z))g_k(\xi)\right)\frac{1}{\phi^m(\xi,z)}d\xi_1\dots d\xi_n\,d\bar{\xi}_I \\
&\quad+\int_{bD}((a_i(\xi,z)T+b_i(\xi,z))(a_j(\xi,z)T \\
&\quad+b_j(\xi,z))g_k(\xi))\frac{1}{\phi^m(\xi,z)}d\xi_1\dots d\xi_n\,d\bar{\xi}_I.
\end{aligned}
$$

Therefore if we let Z denote any one of the following first order differential operators $\frac{\partial}{\partial z_j}$ or $a_j(\xi,z)T+b_j(\xi,z)$ for $j=1,\dots,n$, we see that in general $\frac{\partial^\alpha F_1}{\partial z^\alpha}$ is of the following form for any multiindex α,

$$
\begin{aligned}
\frac{\partial^\alpha}{\partial z^\alpha}F_1(z) &= \underline{2^{|\alpha|-1}}\int_{bD}(Z^{|\alpha|}g_k(\xi))\frac{1}{\phi^m(\xi,z)}d\xi_1\cdots d\xi_n\,d\bar{\xi}_I \qquad (2.14)\\
&= \underline{2^{|\alpha|-1}}\cdot\frac{(-1)^m}{(m-1)!}\int_{bD}((a_0(\xi,z)T \\
&\quad+b_0(\xi,z))^m(Z^{|\alpha|}g_k(\xi)))\log\phi(\xi,z)\,d\xi_1\cdots d\xi_n\,d\bar{\xi}_I
\end{aligned}
$$

where $a_0(\xi,z)=-(T\phi)^{-1}$ and $b_0(\xi,z)=-T((T\phi)^{-1})+h(T\phi)^{-1}$, and underline means there are $2^{|\alpha|-1}$ such terms. Again we see that both $a_0(\xi,z)$ and $b_0(\xi,z)$ are real analytic in z and ξ on $U_z\times U_\xi$. Now if we let W be either Z or $a_0(\xi,z)T+b_0(\xi,z)$, we get

(2.15)

$$
\frac{\partial^\alpha}{\partial z^\alpha}F_1(z)=\underline{2^{|\alpha|-1}}\cdot\frac{(-1)^m}{(m-1)!}\int_{bD}(W^{|\alpha|+m}g_k(\xi))\log\phi(\xi,z)\,d\xi_1\cdots d\xi_n\,d\bar{\xi}_I.
$$

Next we observe that $g_k(\xi)$ can be viewed as a function defined on (z,ξ)-space, i.e., constant in z-variable. Hence if we let X be either the vector field

T or $\frac{\partial}{\partial z_j}$, $j = 1, \dots, n$, then by estimate (2.10) we obtain, for some constant $M > 0$,

$$|X^j g_k(\xi)| \le M(Mk)^j, \text{ provided that } j \le k. \tag{2.16}$$

Also from the real analyticity of both $a_l(\xi, z)$ and $b_l(\xi, z)$ there exists a constant $M_1 > 0$ such that for all $(z, \xi) \in (V_2)_z \times (V_2)_\xi$ we have

$$|D^\beta a_l(\xi, z)| \le M_1 M_1^{|\beta|} \cdot |\beta|!, \; l = 0, 1, 2, \dots, n, \tag{2.17}$$

and

$$|D^\beta b_l(\xi, z)| \le M_1 M_1^{|\beta|} \cdot |\beta|!, \; l = 0, 1, 2, \dots, n, \tag{2.18}$$

where D acts on both variables z and ξ.

We claim that there exists a constant $S > 0$ independent of k, $|\alpha|$ and m such that

$$|W^{|\alpha|+m} g_k(\xi)| \le S(Sk)^{|\alpha|+m}, \tag{2.19}$$

for all $(z, \xi) \in (V_2)_z \times (V_2)_\xi$, provided that $|\alpha| + m \le k$.

Proof of the claim. By induction on i we will show that

$$|X^j W^i g_k| \le M_2 M_3^j M_4^i \cdot k^{j+i} \tag{2.20}$$

holds for all j for some constants M_2, M_3 and M_4, provided that $i + j \le k$. This will prove our claim by setting $j = 0$ and $i = |\alpha| + m$.

First we see that the initial step $i = 0$ follows easily from the construction of g_k, i.e., estimate (2.10) or (2.16). So we assume that (2.20) is true for all j and $i < i_0$. Then we prove the case $i = i_0$ and all j. Put $i_0 = i + 1$ and $j + i + 1 \le k$.

Case 1. If $W^{i+1} = \frac{\partial}{\partial z_l} W^i$, for some $l \in \{1, \dots, n\}$, we have

$$\begin{aligned} X^j W^{i+1} g_k &= X^j \frac{\partial}{\partial z_l} W^i g_k \\ &= X^{j+1} W^i g_k. \end{aligned} \tag{2.21}$$

Hence by induction hypothesis we get

$$\begin{aligned} |X^j W^{i+1} g_k| &\le M_2 M_3^{j+1} M_4^i \cdot k^{i+j+1} \\ &\le M_2 M_3^j M_4^{i+1} \cdot k^{j+i+1}, \end{aligned} \tag{2.22}$$

provided that we choose $M_4 \ge M_3$.

Case 2. If $W^{i+1} = (a_l(\xi, z)T + b_l(\xi, z))W^i$ for some $l = 0, 1, \dots, n$, we get

(2.23)
$$\begin{aligned} X^j W^{i_0} g_k &= X^j (a_l(\xi, z)T + b_l(\xi, z)) W^i g_k \\ &= \pm \sum_{p=0}^{j} \binom{j}{p} (X^p a_l) X^{j-p+1} W^i g_k \pm \sum_{p=0}^{j} \binom{j}{p} (X^p b_l) X^{j-p} W^i g_k \\ &= I + II. \end{aligned}$$

Here underline again means that there are $\binom{j}{p}$ terms of the indicated form. Hence by induction hypothesis we obtain

$$
\begin{aligned}
|I| &\leq \sum_{p=0}^{j} \frac{j!}{p!(j-p)!} M_1 M_1^p \cdot p! \cdot M_2 M_3^{j-p+1} \cdot M_4^i \cdot k^{i+j-p+1} \\
&= M_2 M_3^j M_4^{i+1} \cdot k^{i+j+1} \sum_{p=0}^{j} \frac{j!}{(j-p)!} \cdot \frac{1}{k^p} \cdot \frac{M_1 M_1^p}{M_4 M_3^{p-1}} \\
&\leq M_2 M_3^j M_4^{i+1} \cdot k^{i+j+1} \cdot \left(\frac{M_1 M_3}{M_4} + \sum_{p=1}^{j} \frac{M_1^2}{M_4} \left(\frac{M_1}{M_3} \right)^{p-1} \right) \\
&\leq \frac{1}{2} M_2 M_3^j M_4^{i_0} \cdot k^{j+i_0},
\end{aligned}
$$

provided that $M_3 \geq 2M_1$ and $M_4 \geq \max(8M_1^2, 4M_1 M_3)$.

$$
\begin{aligned}
|II| &\leq \sum_{p=0}^{j} \frac{j!}{p!(j-p)!} M_1 M_1^p \cdot p! \cdot M_2 M_3^{j-p} M_4^i \cdot k^{i+j-p} \\
&= M_2 M_3^j M_4^{i+1} \cdot k^{i+j+1} \cdot \sum_{p=0}^{j} \frac{j!}{(j-p)!} \cdot \frac{1}{k^{p+1}} \cdot \frac{M_1}{M_4} \left(\frac{M_1}{M_3} \right)^p \\
&\leq M_2 M_3^j M_4^{i+1} \cdot k^{i+j+1} \sum_{p=0}^{j} \left(\frac{M_1}{M_4} \right) \left(\frac{M_1}{M_3} \right)^p \\
&\leq \frac{1}{2} M_2 M_3^j M_4^{i_0} \cdot k^{j+i_0},
\end{aligned}
$$

provided that $M_3 \geq 2M_1$ and $M_4 \geq 4M_1$.

This completes the proof of (2.20), and hence our claim (2.19).

So now if we combine the formula (2.15) and Lemma 2.4, let $k = |\alpha| + m$, we see that there exists a constant $S > 0$ such that the following estimate holds uniformly for all $z \in D \cap B(z_0, \varepsilon)$ with $0 < \varepsilon \ll \varepsilon_1$,

$$
\begin{aligned}
\left| \frac{\partial^\alpha}{\partial z^\alpha} F_1(z) \right| &\leq S(S(|\alpha| + m))^{|\alpha|+m} \\
&\leq S(3S)^{|\alpha|+m} \cdot (|\alpha| + m)! \\
&\leq S(3S)^{|\alpha|+m} \cdot (2^m)^{|\alpha|+m} \cdot |\alpha|! \\
&= S(3 \cdot 2^m \cdot S)^m \cdot (3 \cdot 2^m \cdot S)^{|\alpha|} \cdot |\alpha|! \\
&\leq C C^{|\alpha|} |\alpha|!,
\end{aligned}
\tag{2.24}
$$

provided that we choose $C = \max(3 \cdot 2^m \cdot S, S(3 \cdot 2^m \cdot S)^m)$.

This completes the proofs of Theorem 2.5 and Theorem 1.

Acknowledgment

Finally the author would like to thank R. M. Range for fruitful discussions during the preparations of this paper.

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Department of Mathematics, State University of New York at Albany, Albany, New York 12222