

TYCHONOFF REFLECTION IN PRODUCTS AND THE ω -TOPOLOGY ON FUNCTION SPACES

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ABSTRACT. We show that if X is a topological space such that $CR(X)$, the topology on X generated by the cozero sets, is not locally compact, then there is a regular space Y such that $CR(X \times Y) \neq CR(X) \times CR(Y)$. We use the ω -topology on the space of continuous functions $C(X, Y)$ (where ω is an open cover of X) which was defined by Arens and Dugundji in 1950.

In [3], Petr Simon uses the notation $CR(X)$ to denote the topology on X generated by the cozero sets. He asks whether it is true that whenever $CR(X)$ is not locally compact, there exists a regular space Y such that $CR(X \times Y) \neq CR(X) \times CR(Y)$? Another way of putting this (which explains the use of \neq rather than $\not\cong$) is to ask whether it is true that whenever $CR(X)$ is not locally compact, there exists a regular space Y and a cozero set in $X \times Y$ which contains no nonempty product of cozero sets. In this note we answer this question in the affirmative by unearthing an idea of Arens and Dugundji [1]. Arens and Dugundji defined many topologies for the space of continuous functions $C(X, Y)$ from a topological space X to another topological space Y . Among these was the ω -topology, a schema which depended on an open cover ω of X . Although it is somewhat dated to refer to an open cover by a Greek lower case letter, we shall retain their notation. This ω -topology is defined as that generated by sets of the form $\{f \in C(X, Y) : f(F) \subset U\}$ where F is a closed subset of some element of ω and U is an open subset of Y . It should be noted that if ω is a cover of a locally compact Hausdorff space by open sets with compact closure, then the ω -topology is just the compact-open topology. The reason that the ω -topology is rather obscure compared to the compact-open topology is that it is, in general, not regular. This might dissuade one interested in regular, indeed completely regular, spaces from considering such a topology but we will later modify it to be regular, indeed, paracompact

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by using a trick of Oka [2]. We need a couple of facts about the ω -topology which were proved by Arens and Dugundji:

Theorem 1 (Arens, Dugundji [1]). *If ω is an open cover of a topological space X and $C(X, \mathbb{R})$ has the ω -topology, then the evaluation function $g: X \times C(X, \mathbb{R}) \rightarrow \mathbb{R}$ which is defined by $g(x, f) = f(x)$ is continuous. Furthermore, if $C(X, \mathbb{R})$ has any topology which contains the ω -topology, then the evaluation function remains continuous.*

In 1923, Urysohn [5] constructed a Hausdorff space on which all continuous real-valued functions are constant. In 1928, by using Urysohn's idea, Tychonoff [4] constructed a regular space on which all continuous real-valued functions are constant. A lemma which does not seem to have been formally stated but is implicit in these constructions of Urysohn and Tychonoff is the following:

Lemma 1. *If Z is a regular space and Z_0 is a closed subset of Z , then there is a regular space Y which contains Z such that Z is closed in Y , $Z - Z_0$ is open in Y and any continuous real-valued function is constant on $(Y - Z) \cup Z_0$.*

With the use of this lemma we can give a simple proof of the following:

Lemma 2. *If X is a topological space such that $CR(X)$ is not locally compact, then there is a regular space Y such that $CR(X \times Y) \neq CR(X) \times CR(Y)$.*

Proof. Let Ω be the set of all covers of X by cozero sets. Let Z be the free union of $|\Omega|$ many copies of $C(X, \mathbb{R})$, the ω th copy being given the ω -topology. Next, since Z is probably not regular, we make each element of each copy, except the zero function, into an isolated point. Finally we define Z_0 to be the set of all zero functions in Z and use the lemma to get Y . Let $x_0 \in X$ be a point which has no compact neighborhood in $CR(X)$ and let y_0 be any zero function in Y . Let $g: X \times C(X, \mathbb{R}) \rightarrow \mathbb{R}$ be the evaluation function. This function is continuous by Theorem 1. The real-valued function h which we need is that which agrees with the evaluation function on each copy of $X \times C(X, \mathbb{R})$ and is zero everywhere else (this is essentially what Oka was doing). This function is also continuous. We need only show that no rectangular cozero set lies inside $h^{-1}(-1, -1)$ and contains (x_0, y_0) . This is true since if the rectangle is $U \times V$, we know that U does not have compact closure in $CR(X)$ and so that there is an open cover ω in $CR(X)$ such that no finite subcover of ω covers U and, moreover, such that the union of the $CR(X)$ -closures of no finitely many elements of ω covers U . Now V contains y_0 and so it contains every zero function, even that in the ω th copy. No neighborhood of the zero function in the ω th copy can determine more than the behaviour of the function on finitely many closed subsets of elements of ω . Since these elements of ω are contained in finitely many zero sets which do not cover U we can find a point whose behavior has not been dictated and yet which is in U and thus a function which lies inside the rectangle and yet takes that point to 1 which is a contradiction.

The reader should notice that under some circumstances, by interchanging X and Y , a completely regular Y can be found. Of course, if X is completely regular, then equality holds.

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