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TYCHONOFF REFLECTION IN PRODUCTS AND THE ω-TOPOLOGY ON FUNCTION SPACES

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ABSTRACT. We show that if X is a topological space such that $CR(X)$, the topology on X generated by the cozero sets, is not locally compact, then there is a regular space Y such that $CR(X \times Y) \neq CR(X) \times CR(Y)$. We use the ω-topology on the space of continuous functions $C(X, Y)$ (where ω is an open cover of X) which was defined by Arens and Dugundji in 1950.

In [3], Petr Simon uses the notation $CR(X)$ to denote the topology on X generated by the cozero sets. He asks whether it is true that whenever $CR(X)$ is not locally compact, there exists a regular space Y such that $CR(X \times Y) \neq CR(X) \times CR(Y)$? Another way of putting this (which explains the use of $\neq$ rather than $\not\supseteq$) is to ask whether it is true that whenever $CR(X)$ is not locally compact, there exists a regular space Y and a cozero set in $X \times Y$ which contains no nonempty product of cozero sets. In this note we answer this question in the affirmative by unearthing an idea of Arens and Dugundji [1]. Arens and Dugundji defined many topologies for the space of continuous functions $C(X, Y)$ from a topological space X to another topological space Y. Among these was the ω-topology, a schema which depended on an open cover ω of X. Although it is somewhat dated to refer to an open cover by a Greek lower case letter, we shall retain their notation. This ω-topology is defined as that generated by sets of the form $\{f \in C(X, Y): f(F) \subset U\}$ where F is a closed subset of some element of ω and U is an open subset of Y. It should be noted that if ω is a cover of a locally compact Hausdorff space by open sets with compact closure, then the ω-topology is just the compact-open topology. The reason that the ω-topology is rather obscure compared to the compact-open topology is that it is, in general, not regular. This might dissuade one interested in regular, indeed completely regular, spaces from considering such a topology but we will later modify it to be regular, indeed, paracompact

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by using a trick of Oka [2]. We need a couple of facts about the ω-topology which were proved by Arens and Dugundji:

Theorem 1 (Arens, Dugundji [1]). *If ω is an open cover of a topological space X and $C(X, \mathbb{R})$ has the ω-topology, then the evaluation function $g: X \times C(X, \mathbb{R}) \to \mathbb{R}$ which is defined by $g(x, f) = f(x)$ is continuous. Furthermore, if $C(X, \mathbb{R})$ has any topology which contains the ω-topology, then the evaluation function remains continuous.*

In 1923, Urysohn [5] constructed a Hausdorff space on which all continuous real-valued functions are constant. In 1928, by using Urysohn's idea, Tychonoff [4] constructed a regular space on which all continuous real-valued functions are constant. A lemma which does not seem to have been formally stated but is implicit in these constructions of Urysohn and Tychonoff is the following:

Lemma 1. *If Z is a regular space and Z_0 is a closed subset of Z, then there is a regular space Y which contains Z such that Z is closed in Y, $Z - Z_0$ is open in Y and any continuous real-valued function is constant on $(Y - Z) \cup Z_0$.*

With the use of this lemma we can give a simple proof of the following:

Lemma 2. *If X is a topological space such that $CR(X)$ is not locally compact, then there is a regular space Y such that $CR(X \times Y) \neq CR(X) \times CR(Y)$.*

Proof. Let Ω be the set of all covers of X by cozero sets. Let Z be the free union of $|\,\Omega\,|$ many copies of $C(X, \mathbb{R})$, the ωth copy being given the ω-topology. Next, since Z is probably not regular, we make each element of each copy, except the zero function, into an isolated point. Finally we define Z_0 to be the set of all zero functions in Z and use the lemma to get Y. Let $x_0 \in X$ be a point which has no compact neighborhood in $CR(X)$ and let y_0 be any zero function in Y. Let $g: X \times C(X, \mathbb{R}) \to \mathbb{R}$ be the evaluation function. This function is continuous by Theorem 1. The real-valued function h which we need is that which agrees with the evaluation function on each copy of $X \times C(X, \mathbb{R})$ and is zero everywhere else (this is essentially what Oka was doing). This function is also continuous. We need only show that no rectangular cozero set lies inside $h^{-1}(-1, -1)$ and contains (x_0, y_0). This is true since if the rectangle is $U \times V$, we know that U does not have compact closure in $CR(X)$ and so that there is an open cover ω in $CR(X)$ such that no finite subcover of ω covers U and, moreover, such that the union of the $CR(X)$-closures of no finitely many elements of ω covers U. Now V contains y_0 and so it contains every zero function, even that in the ωth copy. No neighborhood of the zero function in the ωth copy can determine more than the behaviour of the function on finitely many closed subsets of elements of ω. Since these elements of ω are contained in finitely many zero sets which do not cover U we can find a point whose behavior has not been dictated and yet which is in U and thus a function which lies inside the rectangle and yet takes that point to 1 which is a contradiction.

The reader should notice that under some circumstances, by interchanging X and Y, a completely regular Y can be found. Of course, if X is completely regular, then equality holds.

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