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A REMARK ON A PAPER BY C. FEFFERMAN

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ABSTRACT. We give a simplified proof of an imbedding theorem by C. Fefferman [3].

The purpose of this paper is to provide a simplified proof of a deep result by C. Fefferman (see [3,1]) concerning the imbedding

$$\int_{\mathbf{R}^n} |u(x)|^p V(x)\,dx \le c \int_{\mathbf{R}^n} |\nabla u(x)|^p\,dx, \qquad \forall u \in C_0^\infty(\mathbf{R}^n). \tag{1.1}$$

In fact (1.1) was proved in [3], for $p = 2$, assuming

$$V \in L^{r,n-2r}(\mathbf{R}^n) \qquad 1 < r \le n/2.$$

Here $L^{r,n-2r}(\mathbf{R}^n) \equiv L^{r,n-2r}$ is the classical Morrey space of the $L^r_{\mathrm{loc}}(\mathbf{R}^n)$ functions such that

$$\sup_{\substack{x\in\mathbf{R}^n\\ \rho>0}} \frac{1}{\rho^{n-2r}} \int_{B(x,\rho)} |V(y)|^r\,dy \equiv \|V\|^r_{r,n-2r} < +\infty$$

where we set $B(x,\rho) = \{y \in \mathbf{R}^n : |x-y| \le \rho\}$.

Our proof rests on the following nice feature of the space $L^{r,n-2r}$: *given $V \in L^{r,n-2r}$ there exists an A_1 weight in the same class majorizing V*. Such a property is not shared by $L^{1,n-2}$ which is well known to be necessary but not sufficient for (1.1) to hold.

Our result is the following:

Theorem. *Let $1 < p < n$, $1 < r \le n/p$, $V \in L^{r,n-pr}$. Then*

$$\int_{\mathbf{R}^n} |u(x)|^p V(x)\,dx \le c\|V\|_{r,n-pr} \int_{\mathbf{R}^n} |\nabla u(x)|^p\,dx, \qquad \forall u \in C_0^\infty(\mathbf{R}^n). \tag{1.2}$$

Here c depends on n and p only.

Proof. To prove (1.2) we assume for a moment that $V \in A_1$ (i.e. $MV(x) \le cV(x)$ a.e. in $\mathbf{R}^n$, where MV is the usual Hardy–Littlewood maximal function). This assumption will be removed later (see Lemma 1).

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For any fixed $u \in C_0^\infty(\mathbf{R}^n)$ let B be a ball such that $u \in C_0^\infty(B)$. Consider the solution z of the Dirichlet problem: $-\Delta z = V$ in B, $z = 0$ on ∂B. Then

$$\int_{\mathbf{R}^n} |u(x)|^p V(x)\,dx \le p \int_{\mathbf{R}^n} |u(x)|^{p-1} |\nabla u(x)| |\nabla z(x)|\,dx. \tag{1.3}$$

Now we observe that using an idea of Hedberg [6] it is easy to prove that the assumption $V \in L^{r,n-pr}$ implies the punctual estimate

$$|\nabla z(x)| \le c(n, r, p)[MV(x)]^{(p-1)/p} \|V\|_{r,n-pr}^{1/p}. \tag{1.4}$$

For the sake of completeness we will prove (1.4) in Lemma 2 below (see also the proof of Theorem 2 in [2]). Substituting in (1.3) we have

$$\begin{aligned}\int_B |u(x)|^p V(x)\,dx &\le c\|V\|_{r,n-pr}^{1/p} \int_B |u(x)|^{p-1} |\nabla u(x)| [MV(x)]^{(p-1)/p}\,dx \\ &\le c\|V\|_{r,n-pr}^{1/p} \left(\int_B |u(x)|^p MV(x)\,dx\right)^{1-1/p} \left(\int_B |\nabla u(x)|^p\,dx\right)^{1/p} \\ &\le c\|V\|_{r,n-pr}^{1/p} \left(\int_B |u(x)|^p V(x)\,dx\right)^{1-1/p} \left(\int_B |\nabla u(x)|^p\,dx\right)^{1/p},\end{aligned}$$

where in the last inequality we used the A_1 assumption on V. The conclusion now follows.

The theorem will be completely proved after we prove the following lemmas.

Lemma 1. *Let* $V \in L^{r,n-pr}$, $1 < p < n$, $1 < r \le n/p$. *Let* $r_1 \colon 1 < r_1 < r$. *Then* $(MV^{r_1})^{1/r_1} \in A_1 \cap L^{r,n-pr}$.

Proof. $(MV^{r_1})^{1/r_1} \in A_1$ clearly (see [5], p. 158). Using the inequality proved in [4] (Lemma 1, p. 111) we have

$$\int_{\mathbf{R}^n} (MV^{r_1})^{r/r_1} \chi(x)\,dx \le c \int_{\mathbf{R}^n} [|V(x)|^{r_1}]^{r/r_1} M\chi(x)\,dx, \tag{1.5}$$

for any χ nonnegative function. Now take as $\chi(x)$ the characteristic function of a ball $B_\rho \equiv B_\rho(x_0)$. We have from (1.5)

$$\begin{aligned}\int_{B_\rho} (MV^{r_1})^{r/r_1}\,dx &\le c\left\{\int_{B_{2\rho}} |V(x)|^r M\chi\,dx + \sum_{k=1}^{+\infty} \int_{B_{2^{k+1}\rho} \setminus B_{2^k\rho}} |V(x)|^r M\chi\,dx\right\} \\ &\le c\left\{\int_{B_{2\rho}} |V(x)|^r M\chi\,dx + \sum_{k=1}^{+\infty} \int_{B_{2^{k+1}\rho} \setminus B_{2^k\rho}} |V(x)|^r \frac{\rho^n}{(|x-x_0|-\rho)^n}\,dx\right\}\end{aligned}$$

Finally

$$\begin{aligned}\int_{B_\rho} [MV^{r_1}]^{r/r_1}\,dx &\le c\|V\|_{r,n-pr}^r \left\{(2\rho)^{n-pr} + \sum_{k=1}^{+\infty} \frac{1}{(2^{k-1})^n} (2^{k+1}\rho)^{n-pr}\right\} \\ &\le c\|V\|_{r,n-pr}^r \rho^{n-pr}.\end{aligned}$$

Lemma 2. *Let* $V \in L^{r,n-pr}$, $1 < p < n$, $1 < r \leq n/p$. *Then*

$$|I_1(V)(x)| = \left| \int_{\mathbf{R}^n} \frac{V(y)}{|x-y|^{n-1}} \, dy \right| \leq c(n,r,p)[MV(x)]^{(p-1)/p} \|V\|_{r,n-pr}^{1/p}.$$

Proof. Let $\rho > 0$. Then

$$I_1(V)(x) = \int_{|x-y|\leq\rho} \frac{V(y)}{|x-y|^{n-1}} \, dy + \int_{|x-y|>\rho} \frac{V(y)}{|x-y|^{n-1}} \, dy \equiv I' + I''.$$

By Hedberg [6] we have

$$|I'| \leq c_n \rho MV.$$

For I'', setting $\sigma = n - (r/2)(p+1)$, we have

$$\begin{aligned} |I''| &\leq \left(\int_{|x-y|>\rho} \frac{|V(y)|^r}{|x-y|^\sigma} \, dy \right)^{1/r} \left(\int_{|x-y|>\rho} |x-y|^{(\sigma/r+1-n)r/(r-1)} \, dy \right)^{1-1/r} \\ &\equiv J' \cdot J''. \end{aligned}$$

For J' using the assumption on V and the trick by Hedberg we have

$$J' \leq c\rho^{(1-p)/2-1)} \|V\|_{r,n-pr}.$$

Finally calculating J'' we get

$$|I_1 V| \leq c' \rho MV + c'' \rho^{1-p} \|V\|_{r,n-pr}.$$

For $\rho = \left(\frac{MV}{\|V\|_{r,n-pr}} \right)^{-1/p}$ we have

$$|I_1 V| \leq c(MV)^{1-1/p} \|V\|_{r,n-pr}^{1/p}.$$

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