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ON INDUCED COVARIANT SYSTEMS

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ABSTRACT. For a closed subgroup H of a locally compact group G, it is shown that a covariant system (G, A) is induced from a covariant system (H, D) if (and only if) there exists a continuous G-equivariant map φ: Prim $A \to G/H$.

Let H be a closed subgroup of a locally compact group G such that H acts strongly continuously by $*$-automorphisms on a C^*-algebra D. Then the induced C^*-algebra $\operatorname{Ind} D$ is defined by

$$\operatorname{Ind} D := \{f \in C^b(G, D) ; f(xh) \overset{h^{-1}}{=} (f(x)) \\ \text{for } x \in G, \quad h \in H \text{ and } \|f(\cdot)\| \in C_0(G/H)\},$$

where $C^b(G, D)$ denotes the space of all D-valued bounded continuous functions on G and $C_0(G/H)$ the space of continuous functions on G/H which vanish at infinity. If we define an action of G on $\operatorname{Ind} D$ by ${}^x f(y) = f(x^{-1}y)$ for all $f \in \operatorname{Ind} D$ and $x, y \in G$, then the pair $(G, \operatorname{Ind} D)$ becomes a covariant system, the so-called induced covariant system of (H, D). It is well known that there is a continuous G-equivariant map φ: $\operatorname{Prim}(\operatorname{Ind} D) \to G/H$, which is defined by $\varphi(J) = xH$ if J contains the ideal $I(x) := \{f \in \operatorname{Ind} D ; f(x) = 0\}$. The following theorem shows that conversely every covariant system (G, A) having this property is isomorphic to a system $(G, \operatorname{Ind} D)$, in the sense that there exists a G-equivariant isomorphism Φ from A onto $\operatorname{Ind} D$.

Theorem. *Suppose that (G, A) is a covariant system, H a closed subgroup of G, and φ: $\operatorname{Prim} A \to G/H$ a continuous G-equivariant map. Furthermore let $I = \cap\{J ; J \in \varphi^{-1}(\{eH\})\}$ and $D = A/I$. Then (G, A) is isomorphic to $(G, \operatorname{Ind} D)$, where the G-equivariant isomorphism Φ: $A \to \operatorname{Ind} D$ is given by $\Phi(a)(x) \overset{x^{-1}}{=} a + I$, for $a \in A$ and $x \in G$.*

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For proving this theorem we need the following lemma.

Lemma. *Suppose that* $\mathscr{F}$ *is a subspace of* $\operatorname{Ind} D$ *such that for every* $x \in G$ *the space* $\mathscr{F}(x) := \{f(x); f \in \mathscr{F}\}$ *is dense in* D. *If* $\mathscr{F}$ *is invariant under multiplication with continuous functions of compact supports on* G/H, *then* $\mathscr{F}$ *is dense in* $\operatorname{Ind} D$.

Proof. Let $g \in \operatorname{Ind} D$ and $\varepsilon > 0$ be given. We choose a compact subset K of G/H such that $\|g(x)\| < \varepsilon/2$ for every $xH \notin K$. If q denotes the quotient map from G onto G/H, then we can find a compact subset C of G such that $q(C) = K$. Now for every $x \in C$ there exist $f_x \in \mathscr{F}$ and a compact neighborhood V_x of x in G such that $\|f_x(y) - g(y)\| < \varepsilon/2$ for every $y \in V_x$. Since C is compact, it is covered by finitely many $V_{x_1}, \dots, V_{x_n}$. Thus K is covered by $W_1, \dots, W_n$, where $W_i = q(V_{x_i})$ for $i = 1, \dots, n$. Now let $\psi_1, \dots, \psi_n$ be a partition of unity for K such that $\operatorname{supp} \psi_i \subseteq W_i$ for every $i \in \{1, \dots, n\}$. Then, for $f(x) = \sum_{i=1}^n \psi_i(xH) f_{x_i}(x)$, we obtain

$$\begin{aligned}
&\|f(x) - g(x)\| \\
&\quad \leq \left\| \sum_{i=1}^n \psi_i(xH)(f_{x_i}(x) - g(x)) \right\| + \left\| \left(1 - \sum_{i=1}^n \psi_i(xH)\right) g(x) \right\| \\
&\quad \leq \sum_{i=1}^n \psi_i(xH) \|f_{x_i}(x) - g(x)\| + \left(1 - \sum_{i=1}^n \psi_i(xH)\right) \|g(x)\| \\
&\quad < \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}$$

for all $x \in G$, which completes the proof.

Proof of the theorem. Firstly we show that $\Phi(a)$ is in fact an element of $\operatorname{Ind} D$, for $D = A/I$. The continuity of $\Phi(a)$ follows from the continuity of the map $x \xrightarrow{x} a$. For $x \in G$ and $h \in H$ we have

$$\Phi(a)(xh) \overset{h^{-1}x^{-1}}{=} a + I \overset{h^{-1}}{=} (\Phi(a)(x)).$$

Now we claim that the function $xH \to \|\Phi(a)(x)\|$ vanishes at infinity. For this let $\varepsilon > 0$ be given, and let us denote by $\widehat{A}$ the space of all equivalence classes of irreducible representations of A. Then, by [1, 3.3.7], the set Q of all $\pi \in \widehat{A}$ with $\|\pi(a)\| \geq \varepsilon$ is a quasicompact subset of $\widehat{A}$. Hence $\tilde{\varphi}(Q)$ is a compact subset of G/H, where $\tilde{\varphi}$ denotes the canonical extension of φ to $\widehat{A}$. Now, for $\pi \in \widehat{A}$ and $x \in G$, let us define the representation ${}^x\pi$ by ${}^x\pi(b) := \pi({}^{x^{-1}}b)$, $b \in A$. Since φ is G-equivariant, it follows that ${}^x\pi \notin Q$ for $\pi \in \widehat{A/I}$ and $xH \notin \tilde{\varphi}(Q)$. Thus

$$\begin{aligned}
\|\Phi(a)(x)\| &= \|{}^{x^{-1}}a + I\| = \sup\{\|\pi({}^{x^{-1}}a)\| ; \pi \in \widehat{A/I}\} \\
&= \sup\{\|{}^x\pi(a)\| ; \pi \in \widehat{A/I}\} \leq \varepsilon
\end{aligned}$$

for every $xH \notin \tilde{\varphi}(Q)$. This proves the claim.

Now it is clear that Φ is a well-defined $*$-homomorphism from A into $\operatorname{Ind} D$, which is isometric since $\bigcap\{{}^xI\,;x \in G\} = \{0\}$. It remains to show that Φ is onto. This follows from the lemma and the trivial fact that

$$\{\Phi(a)(x)\,;a \in A\} = A/I$$

for all $x \in G$, as soon as we have shown that $\Phi(A)$ is invariant under pointwise multiplication with elements of $C_0(G/H)$. For this let $\psi \in C_0(G/H)$ and $a \in A$. Then $\tilde{\psi} = \psi \circ \varphi$ is an element of $C^b(\operatorname{Prim} A)$. Using [2, Theorem 5] in the context of multiplier algebras, we see that there is a unique $z \in Z(\mathscr{M}(A))$, the center of the multiplier algebra $\mathscr{M}(A)$ of A such that $\tilde{\psi}(J)a - za \in J$ for every $J \in \operatorname{Prim} A$. Since $\tilde{\psi}$ is equal to $\psi(xH)$ on $\varphi^{-1}(\{xH\})$ it follows that

$$\psi(xH)a - za \in {}^xI = \bigcap\{J\,;J \in \varphi^{-1}(\{xH\})\}$$

for all $x \in G$. Hence

$$\psi(x\,{}^{x^{-1}}H)\,{}^{x^{-1}}a - {}^{x^{-1}}(za) \in I$$

and therefore

$$\psi(xH)\Phi(a)(x) = \Phi(za)(x)$$

for every $x \in G$. This finishes the proof.

By Proposition 3.1 of [5] the space $\widehat{\operatorname{Ind} D}$ of all equivalence classes of irreducible representations of $\operatorname{Ind} D$ is homeomorphic to $(G \times \widehat{D})/H$, where the action of H on $G \times \widehat{D}$ is given by $h(x\,,\pi) = (xh^{-1}\,,{}^h\pi)$ for $h \in H$ and $(x\,,\pi) \in G \times \widehat{D}$. Note that in [5], $\operatorname{Ind} D$ is denoted by $HC(G\,,D)^\alpha$. Thus, from the theorem we get the following corollary.

Corollary 1. *Let* $(G\,,A)$ *be a covariant system,* H *a closed subgroup of* G, *and* φ: $\operatorname{Prim} A \to G/H$ *a continuous G-equivariant map. Then* $\widehat{A}$ *is homeomorphic to* $(G \times \widehat{A/I})/H$ *where* $I = \bigcap\{J\,;J \in \varphi^{-1}(\{eH\})\}$ *and the action of* H *on* $G \times \widehat{A/I}$ *is defined as above.*

This corollary allows an interesting application to locally compact transformation groups. If H is a closed subgroup of G such that H acts jointly continuously on a locally compact space Y, then H acts on $G \times Y$ by $h(x\,,y) = (xh^{-1}\,,hy)$. The quotient space $(G \times Y)/H$ is usually denoted by $G \times_H Y$. Now G acts on $G \times_H Y$ by inverse left translation of the first component, and it is easily seen that $C_0(G \times_H Y)$ is G-isomorphic to $\operatorname{Ind}(C_0(Y))$. Hence we obtain the following result which shows that Situation 4 and Situation 7 of [7] are the same.

Corollary 2. *Suppose that* $(G\,,M)$ *is a locally compact transformation group,* φ: $M \to G/H$ *a continuous G-equivariant map, and* $Y = \varphi^{-1}(\{eH\})$. *Then* $(G\,,M)$ *is homeomorphic to* $(G\,,G\times_H Y)$ *is the sense that there is a G-equivariant homeomorphism between* $G \times_H Y$ *and* M.

Remark. One can use our theorem, at least in the case of a trivial twisting map, to give a relatively simple proof of [3, Theorem 17], which is the main part of Green's deduction of the Mackey-machine. In fact Green's theorem follows immediately from the Morita equivalence of the crossed product algebras $C^*(H,D)$ and $C^*(G, \operatorname{Ind} D)$. This is a special case of Raeburn's symmetric imprimitivity theorem in [6], but it seems to us that a direct proof of this special case would be a little bit simpler than the proof of Raeburn's more general result. Finally, note that Raeburn also has shown that Green's theorem is a consequence of his symmetric imprimitivity theorem, using substantially the symmetry of his result [6, Special case 2.4].

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