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ON THE SURJECTIVITY CRITERION FOR BUCHSBAUM MODULES

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Dedicated to Professor Hideyuki Matsumura on his 60th birthday

ABSTRACT. Let R be a Cohen–Macaulay local ring with maximal ideal m and suppose that $\dim R \geq 2$. Then R is regular if (and only if) for any Buchsbaum R-module M and for any integer i, $i \neq \dim_R M$, the canonical map $\mathrm{Ext}_R^i(R/m, M) \to H_m^i(M) := \varinjlim_n \mathrm{Ext}_R^i(R/m^n, M)$ is surjective. The hypothesis that R is *Cohen–Macaulay* is not superfluous. Two examples are given.

1. INTRODUCTION

The purpose of this paper is to prove the following

Theorem 1.1. *Let R be a Cohen–Macaulay local ring with maximal ideal $\mathfrak{m}$ and suppose that $\dim R \geq 2$. Then the following two conditions are equivalent.*

(1) *R is a regular local ring.*

(2) *For any Buchsbaum R-module M and for any integer $i \neq \dim_R M$, the canonical map*

$$\mathrm{Ext}_R^i(R/\mathfrak{m}, M) \overset{\varphi_M^i}{\to} H_\mathfrak{m}^i(M) := \varinjlim_n \mathrm{Ext}_R^i(R/\mathfrak{m}^n, M)$$

is surjective.

In the above theorem our contribution is the implication $(2) \Rightarrow (1)$ and the reverse one is due to J. Stückrad [4, Satz 2].

As is well known, Stückrad and Vogel discovered in 1978 a cohomological criterion, so-called now the surjectivity criterion for Buchsbaum modules:

Surjectivity criterion ([4, Satz 2] and [5, p. 732, Theorem 1]). Let M be a finitely generated module over a Noetherian local ring $(R, \mathfrak{m})$. If the canonical

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map $\operatorname{Ext}_R^i(R/\mathfrak{m}, M) \overset{\varphi_M^i}{\to} H_\mathfrak{m}^i(M)$ is surjective for any $i \neq \dim_R M$, M is a Buchsbaum R-module. When R is regular, the converse is also true.

This criterion is general enough and really powerful. In fact, passing to the $\mathfrak{m}$-adic completion $\widehat{R}$ of R and appealing to the structure theorem of Cohen, one may assume the base ring R to be regular; hence a *given R-module M is Buchsbaum if and only if the maps* φ_M^i *are surjective for all* $i \neq \dim_R M$. Comparing with this clear assertion one might feel our Theorem (1.1) somewhat pedantic. However there has been known only one example of Buchsbaum modules M which fail to have the surjectivity of the maps φ_M^i, provided that R is not regular (cf. [4]). On the contrary Theorem (1.1) and its proof claim that any nonregular Cohen–Macaulay local ring R of $d = \dim R \geq 2$ possesses at least one Buchsbaum R-module M of $\dim_R M = d$, for which the canonical map $\operatorname{Ext}_R^1(R/\mathfrak{m}, M) \overset{\varphi_M^1}{\to} H_\mathfrak{m}^1(M)$ is *not* surjective.

The proof of Theorem (1.1) shall be given in the next section. Unfortunately the hypothesis in (1.1) that R is *Cohen–Macaulay* cannot be removed. There exists a nonregular Buchsbaum local ring R of $\dim R = 2$ that satisfies the condition (2) of (1.1) (cf. Proposition (3.2)). We will explore two examples in §3.

Throughout this paper let R stand for a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let $H_\mathfrak{m}^i(\cdot)$ denote the ith local cohomology functor relative to $\mathfrak{m}$.

2. Proof of Theorem 1.1

In this section we assume that R is a Cohen–Macaulay ring of $d = \dim R \geq 2$. We choose a minimal system $x_1, x_2, \dots, x_n$ of generators for the maximal ideal $\mathfrak{m}$ so that the sequence $x_{i_1}, x_{i_2}, \dots, x_{i_d}$ forms a system of parameters of R for any $1 \leq i_1 < i_2 < \cdots < i_d \leq n$. Let

$$0 \longrightarrow L \longrightarrow R^n \xrightarrow{[x_1 x_2 \cdots x_n]} R \longrightarrow R/\mathfrak{m} \longrightarrow 0$$

denote the initial part of a minimal free resolution of $R/\mathfrak{m}$ and let $\{e_i\}_{1 \leq i \leq n}$ be the standard basis of R^n. Then $L \ni f_{ij} := x_i e_j - x_j e_i$ $(1 \leq i < j \leq n)$. We denote by K the R-submodule of L generated by the Koszul relations $\{f_{ij}\}_{1 \leq i < j \leq n}$. Let

$$N = \mathfrak{m}L + K \qquad \text{and} \qquad M = R^n/N.$$

Then we have

Proposition 2.1. *M is a Buchsbaum R-module of* $\dim_R M = d$ *and*

$$\begin{aligned} H_\mathfrak{m}^i(M) &= L/N \qquad (i = 0), \\ &= R/\mathfrak{m} \qquad (i = 1), \\ &= (0) \qquad (i \neq 0, 1, d). \end{aligned}$$

First let us give a proof of (1.1) modulus (2.1). It suffices to prove the implication $(2) \Rightarrow (1)$. Because $\mathrm{Ext}^i_R(R/\mathfrak{m}, M) \xrightarrow{\varphi^i_M} H^i_{\mathfrak{m}}(M)$ is surjective for any $i \neq d$, we get by [5, p. 734, Lemma 6] that the homomorphism

$$j_*\colon \mathrm{Ext}^2_R(R/\mathfrak{m}, L/N) \to \mathrm{Ext}^2_R(R/\mathfrak{m}, M)$$

induced from the imbedding $H^0_{\mathfrak{m}}(M) = L/N \xrightarrow{j} M$ is injective. Let

$$\cdots \to F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 = R^n \xrightarrow{\partial_1 = [x_1 x_2 \cdots x_n]} F_0 = R \to R/\mathfrak{m} \to 0$$

denote a minimal free resolution of $R/\mathfrak{m}$ and recall that the map

$$j_*\colon \mathrm{Ext}^2_R(R/\mathfrak{m}, L/N) \to \mathrm{Ext}^2_R(R/\mathfrak{m}, M)$$

is induced from the following homomorphism

$$\begin{array}{ccccccc}
\cdots \to & \mathrm{Hom}_R(F_1, L/N) & \xrightarrow{\partial_2^*} & \mathrm{Hom}_R(F_2, L/N) & \xrightarrow{\partial_3^*} & \mathrm{Hom}_R(F_3, L/N) & \to \cdots \\
 & \downarrow j_* & & \downarrow j_* & & \downarrow j_* & \\
\cdots \to & \mathrm{Hom}_R(F_1, M) & \xrightarrow[\partial_2^*]{} & \mathrm{Hom}_R(F_2, M) & \xrightarrow[\partial_3^*]{} & \mathrm{Hom}_R(F_3, M) & \to \cdots
\end{array}$$

of complexes. Consider the commutative diagram

$$\begin{array}{ccc}
F_2 & \xrightarrow{\partial_2} & F_1 \\
\partial_2' \downarrow & & \\
L & & \downarrow \varepsilon \\
\tau \downarrow & & \\
L/N & \xrightarrow{j} & M = F_1/N,
\end{array}$$

where ε, τ are the canonical epimorphisms and $\overline{\partial_2'}$ denotes the epimorphism induced from ∂_2. Then the cohomology class $\overline{\tau \circ \partial_2'}$ of $\tau \circ \partial_2'$ is contained in the kernel of $\mathrm{Ext}^2_R(R/\mathfrak{m}, L/N) \xrightarrow{j_*} \mathrm{Ext}^2_R(R/\mathfrak{m}, M)$. Because $\mathrm{Hom}_R(F_2, L/N) = \overline{\mathrm{Ext}^2_R(R/\mathfrak{m}, L/N)}$ and the homomorphism j_* is injective, we see $\tau \circ \partial_2' = \overline{\tau \circ \partial_2'} = 0$ whence $L = N$. As $N = \mathfrak{m}L + K$ by definition, we get $L = K$, that is the module L of the relations of the minimal system $x_1, x_2, \ldots, x_n$ of generators for $\mathfrak{m}$ is generated by the Koszul relations $\{x_i e_j - x_j e_i\}_{1 \le i < j \le n}$. Thus R has to be regular (by an easy Koszul argument: $H_1(x_1, x_2, \ldots, x_n; R) = (0)$ if and only if $x_1, x_2, \ldots, x_n$ is an R-regular sequence).

Proof of Proposition 2.1. By the short exact sequence $0 \to L/N \to M \to \mathfrak{m} \to 0$, we get the second assertion. Hence M is a generalized Cohen–Macaulay R-module, that is the length $l_R(H^i_{\mathfrak{m}}(M))$ of $H^i_{\mathfrak{m}}(M)$ is finite for any $i \neq \dim_R M$, and $I_R(M) = l_R(L/N) + (d-1)$ (cf., e.g., [6,7]). To prove that M is Buchsbaum we need the following lemma.

Lemma 2.2 [7, Proposition 3.2]. *Let R be a Noetherian local ring and let M be a generalized Cohen–Macaulay R-module. Then M is Buchsbaum if and only if the maximal ideal $\mathfrak{m}$ of R contains a system $x_1, x_2, \ldots, x_n$ of generators that satisfies the following condition: For any $1 \le i_1 < i_2 < \cdots < i_s \le n$ $(s = \dim_R M)$, the elements $x_{i_1}, x_{i_2}, \ldots, x_{i_s}$ form a system of parameters for M and one has the equality*

$$l_R(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M) = I_R(M)$$

where $\mathfrak{q} = (x_{i_1}, x_{i_2}, \ldots, x_{i_s})R$.

Let $1 \le i_1 < i_2 < \cdots < i_d \le n$ be integers and put $\mathfrak{q} = (x_{i_1}, x_{i_2}, \ldots, x_{i_d})R$. Then by virtue of (2.2), because our module M is generalized Cohen–Macaulay and $\dim_R M = d$, we have only to see the equality

$$l_R(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M) = l_R(L/N) + (d-1).$$

Recall that the maximal ideal $\mathfrak{m}$ is a Buchsbaum R-module of $I_R(\mathfrak{m}) = d-1$ (cf. [1, Proposition (2.4)]). Then as $e_{\mathfrak{q}}(M) = e_{\mathfrak{q}}(\mathfrak{m})$, we have

$$\begin{aligned} l_R(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M) &= l_R(M/\mathfrak{q}M) - e_{\mathfrak{q}}(\mathfrak{m}) \\ &= l_R(M/\mathfrak{q}M) - [l_R(\mathfrak{m}/\mathfrak{q}\mathfrak{m}) - (d-1)] \\ &= [l_R(M/\mathfrak{q}M) - l_R(\mathfrak{m}/\mathfrak{q}\mathfrak{m})] + (d-1). \end{aligned}$$

Consequently, in order to prove that M is Buchsbaum, it is enough to check the equality

$$l_R(M/\mathfrak{q}M) = l_R(L/N) + l_R(\mathfrak{m}/\mathfrak{q}\mathfrak{m}),$$

that is the sequence

$$0 \to L/N \to M/\mathfrak{q}M \to \mathfrak{m}/\mathfrak{q}\mathfrak{m} \to 0$$

induced from the short exact sequence $0 \to L/N \to M \to \mathfrak{m} \to 0$ remains exact, or equivalently

$$L \cap \mathfrak{q} \cdot R^n \subset N$$

which immediately follows from the next

Claim (2.3). $L \cap \mathfrak{q} \cdot R^n \subset K$.

Proof of Claim 2.3. We may assume $\mathfrak{q} = (x_1, x_2, \ldots, x_d)R$. As

$$\mathfrak{q} \cdot R^n \subset \sum_{i=1}^{d} \mathrm{Re}_i + K,$$

it suffices to show that

$$L \cap \sum_{i=1}^{d} \mathrm{Re}_i \subset K.$$

Let

$$v = \sum_{i=1}^{d} a_i e_i \in L$$

and we have $\sum_{i=1}^{d} a_i x_i = 0$. Because $x_1, x_2, \dots, x_d$ is an R-regular sequence, we see

$$\sum_{i=1}^{d} a_i e_i = \sum_{1 \le i < j \le d} b_{ij}(x_i e_j - x_j e_i)$$

for some $b_{ij} \in R$ which means $v \in K$ as required. This completes the proof of Theorem (1.1) as well as (2.3).

3. Counterexamples

Let M be a Buchsbaum R-module of $\dim_R M = s$ and $\operatorname{depth}_R M = t$. Then it is easy to check that the canonical map $\operatorname{Ext}_R^t(R/\mathfrak{m}, M) \overset{\varphi_M^t}{\to} H_{\mathfrak{m}}^t(M)$ is an isomorphism if $t < s$ (cf. [5, p. 736, Corollary 1.1]). Accordingly, whenever $t < s$ and $H_{\mathfrak{m}}^i(M) = (0)$ for all $i \neq t$, s, the Buchsbaum R-module M enjoys the surjectivity property (2) stated in (1.1). This is the reason why in Theorem (1.1) we have assumed that $\dim R \ge 2$. By the same reason we see that in case $\dim R = 2$, the ring R satisfies the condition (2) of (1.1) if and only if any Buchsbaum R-module M of $\dim_R M = 2$ enjoys the surjectivity property (2) in (1.1).

Proposition 3.1. *Let R be a two-dimensional local integral domain of $e(R) = 1$. Then R satisfies the condition* (2) *of* (1.1).

Proof. By [4, Satz 2] we may assume that R is nonregular. Then R possesses no Buchsbaum module M of $\dim_R M = 2$. In fact, assume the contrary and choose a Buchsbaum R-module M of $\dim_R M = 2$. Then as R is an integral domain, R is contained in the endomorphism algebra $\operatorname{End}_R M$ whence $\widehat{R}$ is a subalgebra of $\operatorname{End}_{\widehat{R}} \widehat{M}$. Let $\mathfrak{P} \in \operatorname{Ass} \widehat{R}$. Then as $\mathfrak{P} \in \operatorname{Ass}_{\widehat{R}} \widehat{M}$ and as $\widehat{M}$ is a Buchsbaum $\widehat{R}$-module, we have either $\dim \widehat{R}/\mathfrak{P} = 2$ or $\mathfrak{P} = \mathfrak{m}\widehat{R}$ (cf. [5, p. 730, Lemma 2]). Of course, since $\operatorname{depth} \widehat{R} > 0$, we get $\mathfrak{P} \neq \mathfrak{m}\widehat{R}$ and so $\dim \widehat{R}/\mathfrak{P} = 2$ for any $\mathfrak{P} \in \operatorname{Ass} \widehat{R}$. Hence R is unmixed, which implies by [3, (40.6) Theorem] that R is a regular local ring because $e(R) = 1$ by our assumption—this contradicts the choice of R. Thus R possesses no Buchsbaum module M of $\dim_R M = 2$.

In his famous book [3, p. 203, Example 2] Nagata constructed a two-dimensional nonregular local integral domain R of $e(R) = 1$. His example asserts by (3.1) that the hypothesis in (1.1) that R is *Cohen–Macaulay* is not superfluous.

Proposition 3.2. *Let S be a three-dimensional regular local ring with maximal ideal $\mathfrak{n}$. Let $X \in \mathfrak{n} \setminus \mathfrak{n}^2$ and let I be a proper ideal in S of $ht_S I \ge 2$. Then the ring $R = S/XI$ satisfies the condition* (2) *in* (1.1).

Proof. Let $\mathfrak{P} = XR$. Then R possesses exactly three isomorphism classes of indecomposable Buchsbaum R-modules M of $\dim_R M (= \dim R) = 2$ and the

R-modules

$$M_0 = R/\mathfrak{m}\mathfrak{P}, \qquad M_1 = \mathfrak{m}/\mathfrak{P}, \qquad \text{and } M_2 = R/\mathfrak{P}$$

are the representatives of them (cf. [2, Theorem (3.1)]). Since $H^i_{\mathfrak{m}}(M_j) = 0$ if $i \neq j, 2$, the R-modules $M_j (j = 0, 1, 2)$ enjoy the property (2) in (1.1). Because any Buchsbaum R-module M of $\dim_R M = 2$ is isomorphic to a direct sum of M_j's together with a vector space over $R/\mathfrak{m}$, we see that M has the required property (2) stated in (1.1). Thus R satisfies the condition (2) of (1.1).

In the above proposition, if we choose $I = \mathfrak{n}$, $R = S/X\mathfrak{n}$ is a nonregular Buchsbaum ring. This example shows that the hypothesis in (1.1) that R is *Cohen–Macaulay* cannot be replaced by the weaker one that R is *Buchsbaum*.

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