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CLOSURE OF INVERTIBLE OPERATORS ON A HILBERT SPACE

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ABSTRACT. Although most of the characterizations of the closure of the invertible operators on a separable Hilbert space do not extend to a nonseparable Hilbert space, this note gives a characterization for an arbitrary Hilbert space that generalizes the separable case in a natural way. The new concept of essential nullity, which facilitates this characterization, should find other applications.

1. INTRODUCTION

This research is motivated by a theorem proved by Robin Harte in [6]. In order to state that result we recall the following definitions. An element a of the Banach algebra $\mathscr{A}$ is said to be regular provided there is an element $b \in \mathscr{A}$ such that $a = aba$. We say that a is decomposably regular provided the b in the preceding equation can be chosen to be an invertible element of $\mathscr{A}$. Harte's theorem asserts that a regular element a belongs to the closure of the invertible elements if and only if it is decomposably regular.

Much of the interest in Harte's theorem arises from its application to the Banach algebra $\mathscr{B}(H)$ of (bounded linear) operators on the Hilbert space H. For $T \in \mathscr{B}(H)$ we define $\operatorname{nul} T$ and $\operatorname{def} T$ to be the cardinal numbers $\dim \ker T$ and $\dim \ker T^*$, respectively. It follows from the work of Atkinson [1] that T is regular if and only if T has closed range. We shall prove in the next section that a regular operator T is decomposably regular if and only if $\operatorname{nul} T = \operatorname{def} T$. Thus, it follows from Harte's theorem that an operator T with closed range belongs to the closure of the invertible operators $\mathscr{G}$ if and only if $\operatorname{nul} T = \operatorname{def} T$. This characterization does not require the underlying Hilbert space to be separable.

When H is separable there is a nice characterization of the closure of $\mathscr{G}$ that is immediate from the main theorem of [3]. The operator T belongs to the closure of $\mathscr{G}$ if and only if $\operatorname{nul} T = \operatorname{def} T$ or the range of T is not closed. Regardless of whether H is separable or not, the closure of $\mathscr{G}$ is characterized in [4] but the characterization uses W^* algebra existence theorems and

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concepts that are difficult to interpret concretely. This note will establish a characterization that is analogous to Harte's theorem and is easy to understand concretely.

Let $U|T|$ be the usual polar factorization of T and let $E(\cdot)$ be the spectral measure for the nonnegative operator $|T|$. We define $\operatorname{ess\,nul} T$ by the equation

$$\operatorname{ess\,nul} T = \inf\{\dim E([0,\varepsilon))H : \varepsilon > 0\}$$

and we define $\operatorname{ess\,def} T$ by

$$\operatorname{ess\,def} T = \operatorname{ess\,nul} T^* .$$

2. Main results

First we prove the previously mentioned proposition that facilitates the application of Harte's theorem to the Hilbert space case.

1. **Proposition.** *Suppose $T \in \mathscr{B}(H)$ has closed range. The operator T is decomposably regular if and only if* $\operatorname{nul} T = \operatorname{def} T$.

Proof. If $\operatorname{nul} T = \operatorname{def} T$ then there is a linear isometry V of $(TH)^\perp$ onto $\ker T$. Let B coincide with V on $(TH)^\perp$ and on TH let it be the inverse of T restricted to $(\ker T)^\perp$. Then $TBT = T$ holds.

Assume that $TBT = T$ holds for an invertible operator B. Since TB is idempotent and $TH = TBH$, we see that $H = TH \oplus \ker TB$. It follows from this equation that

$$\dim\ker T^* = \dim(TH)^\perp = \dim\ker TB = \dim\ker T .$$

We shall need the next lemma in the proof of the main theorem.

2. **Lemma.** *Let $T = U|T|$ be the usual polar factorization and let V be the isometry obtained by restricting U to the closure of the range of T^*, denoted $(T^*H)^-$, and considering $(TH)^-$ to be the range of V. Let $E(\cdot)$ and $F(\cdot)$ denote the spectral measures of $|T|$ and $|T^*|$, respectively. Then on the obvious subspaces we have $|T^*| = V|T|V^*$ and $F(\mathscr{I}) = VE(\mathscr{I})V^*$ for any interval $\mathscr{I}$ contained in* $(0, \infty)$.

Proof. Note that $TT^* = U|T|^2U^* = U(T^*T)U^*$. Since the square root of an operator is the limit of polynomials in the operator (see [5, Problem 95]), we see that $V|T|V^* = |T^*|$. The final assertion follows from the equations

$$|T^*| = V|T|V^* = V\left(\int_{\mathscr{I}} t\,dE(t)\right)V^*,$$

$$|T^*| = \int_{\mathscr{I}} t\,dVE(t)V^*$$

and the fact that the last equation characterizes $F(\cdot)$.

Now we can state and prove our main result.

3. **Theorem.** *The operator* T *belongs to the closure of the invertible operators* $\mathcal{G}$ *if and only if* $\operatorname{ess\,nul} T = \operatorname{ess\,def} T$.

Proof. Because $\dim E([0,\varepsilon))H$ is a discrete valued nondecreasing function of ε, we conclude that there is a positive γ such that $\dim E([0,\varepsilon))H = \operatorname{ess\,nul} T$ for $0 < \varepsilon \le \gamma$. Similarly we get $\dim F([0,\varepsilon))H = \operatorname{ess\,def} T$ for $0 < \varepsilon \le \gamma$ and so we assume that $\dim E([0,\varepsilon))H = \dim F([0,\varepsilon))H$ for $0 < \varepsilon \le \gamma$. Define $R(\varepsilon)$ to agree with $|T|$ on $E([\varepsilon,\infty))H$ and to agree with εI on $E([0,\varepsilon))H$; define $U(\varepsilon)$ to agree with U on $E([\varepsilon,\infty))H$ and on $E([0,\varepsilon))H$ let it agree with V, where V is an isometry of $E([0,\varepsilon))H$ onto $F([0,\varepsilon))H$. Using the final conclusion of the lemma and the fact that

$$UE([0,\infty))H = U(|T|H)^- = (TH)^- = (|T^*|H)^-,$$

we see that $U(\varepsilon)$ is one-to-one and onto; thus, $U(\varepsilon)$ is invertible. Clearly $R(\varepsilon)$ is invertible and it is straightforward to see that

$$\|UR - U(\varepsilon)R(\varepsilon)\| \le 2\varepsilon.$$

This proves that T belongs to $\mathcal{G}^-$.

Assume that A_j is a sequence of invertible operators such that $\|T - A_j\| \to 0$ and, for the sake of a contradiction assume that $\operatorname{ess\,nul} T \ne \operatorname{ess\,def} T$. Replace T with T^*, if necessary, so that $\operatorname{ess\,nul} T > \operatorname{ess\,def} T$. Note that $\|T^*T - A_j^*A_j\| \to 0$. By the continuity of square roots (see [2, Theorem 2]), we conclude that $\||T| - |A_j|\| \to 0$.

Choose $\gamma > 0$ such that

$$\dim E([0,\varepsilon))H = \operatorname{ess\,nul} T, \qquad \dim F([0,\varepsilon))H = \operatorname{ess\,def} T$$

for $0 < \varepsilon \le \gamma$. Let $H_\gamma = E([\gamma,\infty))H$ and define B on H_γ to agree with the inverse of $|T|$ restricted to H_γ, denoted $(|T||H_\gamma)^{-1}$; let B be zero on $(H_\gamma)^\perp$. Note that $TB = UE([\gamma,\infty))$ and that $U|A_j|(B + 1/j)$ converges to TB. Let the polar factorization of $U|A_j|(B + 1/j)$ be denoted W_jR_j where $R_j \ge 0$; note that W_j is unitary and $\|R_j - W_j^*UE([\gamma,\infty))\| \to 0$. Choose j sufficiently large that

$$(*) \qquad \|R_j - W_j^*UE([\gamma,\infty))\| < \frac{1}{3}.$$

Because the dimension of $\ker W_j^*UE([\gamma,\infty)) = E([0,\gamma))H$ exceeds $\dim W_j^*F([0,\gamma))H$ there exists a unit vector f in $E([0,\gamma))H$ that is orthogonal to $W_j^*F([0,\gamma))H$. This follows from the observation that $F([0,\gamma))W_j$ restricted to $E([0,\gamma))H$ must have nontrivial kernel. The next equations follow primarily from the lemma:

$$\begin{aligned}(W_j^*UE([\gamma,\infty)))^* &= (W_j^*UE([\gamma,\infty))U^*U)^* = (W_j^*F([\gamma,\infty))U)^* \\ &= U^*F([\gamma,\infty))W_j.\end{aligned}$$

Thus, the kernel of $(W_j^* UE([\gamma,\infty)))^*$ is $W_j^* F([0,\gamma))H$. From the inequality $(*)$ and the fact that $U^* F([\gamma,\infty))W_j$ is a partial isometry we get

$$\frac{1}{3} > \|R_j f - W_j^* UE([\gamma,\infty))f\| = \|R_j f\|$$

and

$$\frac{1}{3} > \|R_j f - U^* F([\gamma,\infty))W_j f\| \geq \|U^* F([\gamma,\infty))W_j f\| - \|R_j f\| \geq \frac{2}{3}.$$

This contradiction proves that $\operatorname{ess\,nul} T = \operatorname{ess\,def} T$.

Note that Theorem 3 extends Harte's theorem in the Hilbert space case since $\operatorname{ess\,nul} T = \operatorname{nul} T$ and $\operatorname{ess\,def} T = \operatorname{def} T$ when T has closed range.

The next lemma provides an alternative characterization of $\operatorname{ess\,nul} T$. We shall use the result in the proof of Theorem 5.

4. **Lemma.** *Let $\varepsilon > 0$ and cardinal number β be given. We have $\dim E([0,\varepsilon))H = \beta$ if and only if the inequality $\|T|H_\varepsilon\| < \varepsilon$ implies that $\dim H_\varepsilon \leq \beta$ and equality is achieved in the last inequality for some choice of H_ε.*

Proof. If $\dim E([0,\varepsilon))H = \beta$ then we choose H_ε to be $E([0,\varepsilon))H$ and we need only prove that there is no subspace H_0 with dimension higher than β and $\|T|H_0\| < \varepsilon$. For the sake of a contradiction assume that $\dim H_0 = \alpha > \beta$. Since

$$\dim E([0,\varepsilon))H_0 \leq \dim E([0,\varepsilon))H = \beta,$$

there is a nonzero vector $f \in H_0$ such that $E([0,\varepsilon))f = 0$. Thus, $f = E([\varepsilon,\infty))f$ and

$$\||T|f\| = \||T|E([\varepsilon,\infty))f\| \geq \varepsilon\|f\|$$

which is a contradiction.

Suppose there is a subspace H_ε of dimension β such that $\|T|H_\varepsilon\| < \varepsilon$ and no subspace of higher dimension has this property. If $\dim E([0,\varepsilon))H = \gamma < \beta$ then the argument in the preceding paragraph leads to a contradiction. It is not possible for $\gamma > \beta$ since H_ε can be chosen to be $E([0,\varepsilon))H$. This proves the lemma.

The next theorem shows that $\operatorname{ess\,nul} T$ has a lower semicontinuous property.

5. **Theorem.** *If $\|T - A_j\| \to 0$ and $\alpha = \liminf \operatorname{ess\,nul} A_j$ then $\operatorname{ess\,nul} T \geq \alpha$.*

Proof. Replace A_j with a subsequence such that $\lim \operatorname{ess\,nul} A_j = \alpha$. Let $G_j(\cdot)$ denote the spectral measure for $|A_j|$. Since the values of $\operatorname{ess\,nul} A_j$ are discrete, it follows that $\dim G_j([0,\varepsilon))H = \operatorname{ess\,nul} A_j$ for positive ε sufficiently small and that $\operatorname{ess\,nul} A_j = \alpha$ for j sufficiently large. Let $\varepsilon > 0$ be given. Choose j sufficiently large that $\|T - A_j\| < \varepsilon/2$ and $\operatorname{ess\,nul} A_j = \alpha$. Then choose $\gamma > 0$ sufficiently small that $\gamma < \varepsilon/2$ and $\dim G_j([0,\gamma))H = \operatorname{ess\,nul} A_j = \alpha$. Note that

$$\|T|G_j([0,\gamma))H\| \leq \|A_j|G_j([0,\gamma))H\| + \|T - A_j\| < \varepsilon.$$

By the preceding lemma we conclude that $\dim E([0,\varepsilon))H \geq \dim G_j([0,\gamma))H = \alpha$.

3. Remarks

In this section we note that a certain characterization of the closure of the invertible operators $\mathcal{G}^-$ on a separable Hilbert space does not hold more generally. Izumino shows in [7, Corollary to Theorem 2] that $\mathcal{G}^-$ is the set of all compact perturbations of the operators T with $\operatorname{nul} T = \operatorname{def} T$. The next result shows that Izumino's characterization does not extend to the nonseparable case. We say that T has index equal to zero, written $\operatorname{ind} T = 0$, provided $\operatorname{nul} T = \operatorname{def} T$.

6. **Theorem.** *The set of operators $T + K$ with $\operatorname{ind} T = 0$ and K a compact operator is a subset of the closure of the invertible operators $\mathcal{G}^-$ but sometimes it is a proper subset.*

Proof. In order to show that $(T + K) \in \mathcal{G}^-$ we use Izumino's argument. Let $T = U|T|$ be a polar factorization with U unitary. For each positive integer j we see that $(|T| + 1/j + U^*K)$ is a Fredholm operator with index 0 since it is a compact perturbation of an invertible operator. Thus, $U(|T| + 1/j + U^*K)$ is a sequence of Fredholm operators with index 0 that converges to $T + K$. It follows from Theorem 3 or Harte's theorem that any Fredholm operator with index 0 is the limit of invertible operators. Thus, $(T + K) \in \mathcal{G}^-$.

Now we describe an operator B in $\mathcal{G}^-$ that cannot be represented as $T + K$ with $\operatorname{ind} T = 0$ and K compact. Let R be multiplication by the independent variable on $H = L^2(d\mu)$ where μ is the counting measure on $[0, 1]$—i.e., $\mu(\mathcal{S})$ is the cardinality of $\mathcal{S}$. Note that the Schauder dimension of H is $2^{\aleph_0}$. Let H_0 be the subspace $H\chi_{[0,.5]}$ where $\chi_{[0,.5]}$ is the characteristic function of the closed interval $[0, .5]$. Let U be an isometry of H onto H_0 and let $B = UR$. Note that $\operatorname{nul} B = 0$, $\operatorname{def} B = 2^{\aleph_0}$ and, $\operatorname{ess\,nul} B = 2^{\aleph_0} = \operatorname{ess\,def} B$. By Theorem 3 it follows that $B \in \mathcal{G}^-$.

No compact perturbation of B can have index zero because $\operatorname{nul}(B+K) \le \aleph_0$ and $\operatorname{def}(B + K) = 2^{\aleph_0}$ for any compact operator K. Both observations follow from the fact that $\dim(\ker K)^\perp \le \aleph_0$ for any compact operator K. This last assertion follows from the Riesz-Schauder theory for compact operators. See [8], for example.

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