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THE MATHERON REPRESENTATION THEOREM FOR GRAY-SCALE MORPHOLOGICAL FILTERS

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ABSTRACT. We present a new proof of the Matheron representation theorem for gray-scale morphological filters, without using either the representation theorem for subsets of the plane or the umbra transform.

1. INTRODUCTION

An important theorem in mathematical morphology is the Matheron representation theorem, which for subsets of R^2 may be stated as follows [2, Chapter 5]: if Ψ is an increasing translation-invariant mapping between subsets of R^2, then for any subset A of R^2 we have

$$\Psi(A) = \bigcup_{B \in \operatorname{Ker} \Psi} \mathscr{E}(A, B),$$

where $\mathscr{E}(A, B)$ denotes the erosion of the set A by the set B, and $\operatorname{Ker}\Psi$ (the kernel of Ψ) is the collection of all subsets B of R^2 such that $\Psi(B)$ contains the origin.

In [2, Chapter 7], the Matheron representation theorem has been extended to the case of gray-scale morphological filters. In their proof, however, the authors use the representation theorem for subsets of R^2 and also the so-called "umbra transform". In the following sections, we give a proof of the Matheron representation theorem for gray-scale morphological filters without using either the theorem for R^2 or the umbra transform; moreover, the theorem for subsets of R^2 is just a special case of our general theorem.

2. DEFINITIONS AND NOTATIONS

Let G be a (not necessarily commutative) group with a multiplicative group operation and identity e; the inverse of $x \in G$ is denoted by x^{-1}.

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When V is a subgroup of the additive group R of real numbers, $G \times V$ is also a group for the following operation: for $x \in G$, $y \in G$, $r \in V$, and $s \in V$, we define

$$(x, r) \cdot (y, s) = (xy, r + s).$$

For the case of mathematical morphology, G may be thought of as $(R^2, +)$ or $(Z^2, +)$, while for V we take $V = R$, $V = Z$ or $V = \{0\}$, where Z is the set of integers. For $V = \{0\}$, $G \times V$ can be identified in the usual manner with G.

We denote by $\mathcal{S}$ the set of bounded functions f defined on a subset D_f (also written as $D(f)$) of G and with values in V. $\mathcal{S}^*$ is the set of functions defined on or subset of G and with values in $V \cup \{+\infty\}$. We identify f with its graph $G(f)$; hence, for $f \in \mathcal{S}$ we have $f \equiv G(f) = \{(x, f(x)): x \in D_f\}$, which is a subset of $G \times V$. When $V = \{0\}$, then $f = \{(x, 0): x \in D_f\} \equiv D_f$, which shows that in that case we may consider $\mathcal{S}$ to be the set 2^G.

For f and g in $\mathcal{S}$, the notation $f \ll g$ means that $D_f \subset D_g$ and $f(x) \leqq g(x) \forall x \in D_f$; specifically, for $V = \{0\}$, we have $f \ll g \Leftrightarrow D_f \subset D_g$. When $(b, r) \in G \times V$ and $f \in \mathcal{S}$, the left-translate ${}_{(b,r)}f$ is defined by $({}_{(b,r)}f)(x) = f(b^{-1}x) + r$. In particular, for $V = \{0\}$, we have ${}_{(b,0)}f \equiv D_{(b,0)}f = (b, 0) \cdot D_f$, where in the last term we identify D_f with $(D_f, 0)$.

According to the terminology in [2], a mapping $\Psi: \mathcal{S} \to \mathcal{S}^*$ is called *increasing* if $f \ll g$ implies $\Psi(f) \ll \Psi(g)$ for all f, $g \in \mathcal{S}$; Ψ is called *left translation invariant* if $\Psi({}_{(b,r)}f) = {}_{(b,r)}\Psi(f)$ for all $(b, r) \in G \times V$ and all $f \in \mathcal{S}$. An increasing left translation invariant mapping $\Psi: \mathcal{S} \to \mathcal{S}^*$ is called a *morphological filter*. The *kernel* $\operatorname{Ker}\Psi$ of such filter is defined by $\operatorname{Ker}\Psi = \{f \in \mathcal{S}: \Psi(f)(e) \geqq 0\}$. For $V = \{0\}$, this leads to $\operatorname{Ker}\Psi = \{D_f: e \in \Psi(D_f)\}$, which corresponds to the usual definition for mappings between subsets of R^2. The final notation is the *erosion* $\mathcal{E}(f, g)$ of a function f in $\mathcal{S}$ by a function g in $\mathcal{S}$; again, it is a function in $\mathcal{S}$ defined as

$$\mathcal{E}(f, g) = \{(x, t) \in G \times V: xD_g \subset D_f, t = \sup\{s \in V: g(x^{-1}\cdot) + s \leqq f(\cdot)\}.$$

This definition may be found in [2], and also in [1], where we gave a unifying theory for the morphological operations dilation, erosion and opening for gray-scale images; in particular, for $V = \{0\}$ and $G = R^2$, we are again led to the erosion of two subsets of R^2. The only properties we need in the sequel, and which may readily be derived from the definition, are

$$\mathcal{E}({}_{(b,r)}f, g) = {}_{(b,r)}\mathcal{E}(f, g),$$
$$f_1 \ll f_2 \Rightarrow \mathcal{E}(f_1, g) \ll \mathcal{E}(f_2, g).$$

3. The Matheron representation theorem

Proposition 1. *Let* Ψ_1 *and* Ψ_2 *be morphological filters. Then*

$$\operatorname{Ker}\Psi_1 \subset \operatorname{Ker}\Psi_2 \Leftrightarrow \Psi_1(f) \ll \Psi_2(f), \qquad \forall f \in \mathscr{S}.$$

Proof. Suppose $\Psi_1(f) \ll \Psi_2(f)$ for all $f \in \mathscr{S}$. Given $f \in \operatorname{Ker}\Psi_1$, then $\Psi_1(f)(e) \geqq 0$. From our assumption we have $D(\Psi_1(f)) \subset D(\Psi_2(f))$ and $\Psi_1(f)(x) \leqq \Psi_2(f)(x)$ for all $x \in D(\Psi_1(f))$. Hence, it follows that $\Psi_2(f)(e)$ is defined and $\Psi_1(f)(e) \leqq \Psi_2(f)(e)$; therefore $f \in \operatorname{Ker}\Psi_2$.

Conversely, suppose $\operatorname{Ker}\Psi_1 \subset \operatorname{Ker}\Psi_2$. We must show that for each f in $\mathscr{S}$, $D(\Psi_1(f)) \subset D(\Psi_2(f))$, and for each x in $D(\Psi_1(f))$, $\Psi_1(f)(x) \leqq \Psi_2(f)(x)$.

(i) If it is not true that $D(\Psi_1(f)) \subset D(\Psi_2(f))$ for all f in $\mathscr{S}$, then there exists $f \in \mathscr{S}$ and $x \in D(\Psi_1(f))$ such that $x \notin D(\Psi_2(f))$. Suppose $\Psi_1(f)(x) = a \in V$, and consider the function ${}_{(x^{-1},-a)}f$. Then

$$\Psi_1({}_{(x^{-1},-a)}f)(e) = \Psi_1(f)(x) - a = 0,$$

which means that ${}_{(x^{-1},-a)}f$ belongs to $\operatorname{Ker}\Psi_1$; however, $\Psi_2({}_{(x^{-1},-a)}f)(e)$ is not defined since $x \notin D(\Psi_2(f))$, and so ${}_{(x^{-1},-a)}f$ is not an element of $\operatorname{Ker}\Psi_2$. This is a contradiction.

(ii) Suppose there exists $f \in \mathscr{S}$ and $x \in D(\Psi_1(f))$ such that $\Psi_1(f)(x) > \Psi_2(f)(x)$. If $\Psi_1(f)(x) = a \in V$, consider again the function ${}_{(x^{-1},-a)}f$; then $\Psi_1({}_{(x^{-1},-a)}f)(e) = 0$, while $\Psi_2({}_{(x^{-1},-a)}f)(e) < 0$. This is again a contradiction.

(When $\Psi_1(f)(u) = +\infty$ in (i) or (ii), the proof is easily adapted).

Corollary 1. *When* Ψ_1 *and* Ψ_2 *are morphological filters, then*

$$\Psi_1 = \Psi_2 \Leftrightarrow \operatorname{Ker}\Psi_1 = \operatorname{Ker}\Psi_2.$$

Given a fixed function g in $\mathscr{S}$, we define the mapping Ψ_g on $\mathscr{S}$ by means of

$$\Psi_g(f) = \mathscr{E}(f, g), \qquad f \in \mathscr{S}.$$

Proposition 2. (i) Ψ_g *is a morphological filter.*

(ii) $\operatorname{Ker}\Psi_g = \{f : g \ll f\}$.

Proof. (i) This follows immediately from the properties of erosion, as mentioned at the end of §2.

(ii) $f \in \operatorname{Ker}\Psi_g$ iff $\Psi_g(f)(e) \geqq 0$ iff $\mathscr{E}(f, g)(e) \geqq 0$.

Now $e \in D(\mathscr{E}(f, g))$ iff $eD_g \subset D_f$, which is fulfilled as soon as $g \ll f$. Also $\mathscr{E}(f, g)(e) \geqq 0$ iff $\sup\{s \in V : s \leqq f(z) - g(ez),\ \forall z \in D_g\} \geqq 0$, which is true iff $g(z) \leqq f(z)$, $\forall z \in D_g$.

Lemma 1. *Let* Ψ *be a morphological filter. Let* Ψ_1 *be the mapping* $\mathscr{S} \to \mathscr{S}^*$, *defined as*

$$D(\Psi_1(f)) = \bigcup_{g \in \operatorname{Ker}\Psi} D(\mathscr{E}(f, g))$$

$$(\Psi_1(f))(x) = \sup \mathscr{E}\{(f, g)(x) : g \in \operatorname{Ker}\Psi \textit{ such that } x \in D(\mathscr{E}(f, g))\}.$$

Then Ψ_1 *is a morphological filter.*

Proof. (i) We first investigate the left translation invariance of Ψ_1. Let $(b,r) \in G \times V$, $f \in \mathscr{S}$. Then, since $D({}_{(b,r)}f) = bD_f$ (for the group operation in G), we have

$$\begin{aligned} D(\Psi_1({}_{(b,r)}f) &= \bigcup_{g\in \operatorname{Ker}\Psi} D(\mathscr{E}({}_{(b,r)}f, g)) \\ &= \bigcup_{g\in \operatorname{Ker}\Psi} D({}_{(b,r)}\mathscr{E}(f, g)) \\ &= \bigcup_{g\in \operatorname{Ker}\Psi} bD(\mathscr{E}(f, g)), \end{aligned}$$

and also

$$D({}_{(b,r)}\Psi_1(f)) = bD(\Psi_1(f)) = \bigcup_{g\in \operatorname{Ker}\Psi} bD(\mathscr{E}(f, g)).$$

Moreover,

$$\begin{aligned} \Psi_1({}_{(b,r)}f)(x) &= \sup\{\mathscr{E}({}_{(\mathrm{b,r})}\mathrm{f}, \mathrm{g})(\mathrm{x}) \colon g \in \operatorname{Ker}\Psi \text{ such that } x \in D(\mathscr{E}({}_{(b,r)}f, g))\} \\ &= \sup\{{}_{(\mathrm{b,r})}\mathscr{E}(\mathrm{f}, \mathrm{g})(\mathrm{x}) \colon g \in \operatorname{Ker}\Psi \text{ such that } x \in bD(\mathscr{E}(f, g))\} \\ &= \sup\{(\mathscr{E}(\mathrm{f}, \mathrm{g})(\mathrm{b}^{-1}\mathrm{x}) + \mathrm{r}) \colon g \in \operatorname{Ker}\Psi \text{ such that } b^{-1}x \in D(\mathscr{E}(f, g))\}, \end{aligned}$$

which is exactly the value of ${}_{(b,r)}\Psi_1(f)(x)$.

(ii) To show that Ψ_1 is also increasing, we have to prove that $f \ll h$ implies $D(\Psi_1(f)) \subset D(\Psi_1(h))$ and $\Psi_1(f)(x) \leqq \Psi_1(h)(x)$ for all $x \in D(\Psi_1(f))$.

This is almost obvious from the definition of Ψ_1, due to the fact that $D(\mathscr{E}(f, g)) \subset D(\mathscr{E}(h, g))$ and that $\mathscr{E}(f, g)(x) \leq \mathscr{E}(h, g)(x)$.

Theorem 1. The Matheron representation theorem. *Let* Ψ *be a morphological filter. Then for each* f *in* $\mathscr{S}$, $\Psi(f)$ *is the function defined as*

$$D(\Psi(f)) = \bigcup_{g\in \operatorname{Ker}\Psi} D(\mathscr{E}(f, g)), \textit{ and}$$

$$\Psi(f)(x) = \sup\{\mathscr{E}(\mathrm{f}, \mathrm{g})(\mathrm{x}) \colon g \in \operatorname{Ker}\Psi \textit{ such that } x \in D(\mathscr{E}(f, g))\}.$$

Proof. We first remark that, for $g \in \operatorname{Ker}\Psi$ and $g \ll h$, $h \in \operatorname{Ker}\Psi$ also. Hence, $\operatorname{Ker}\Psi \supset \bigcup_{g\in \operatorname{Ker}\Psi} \{h \colon g \ll h\}$. But it is trivial that any g in $\operatorname{Ker}\Psi$ also belongs to the set $\{h \colon g \ll h\}$. This leads to

$$\operatorname{Ker}\Psi = \bigcup_{g\in \operatorname{Ker}\Psi} \{h \colon g \ll h\}.$$

Taking into account the function Ψ_1 introduced in Lemma 1, the theorem will be proved if, according to Corollary 1, we show that $\operatorname{Ker}\Psi = \operatorname{Ker}\Psi_1$.

First, suppose that $f \in \operatorname{Ker}\Psi_1$; then $e \in D(\Psi_1(f))$ and $\Psi_1(f)(e) \geqq 0$. From the first conclusion and the definition of Ψ_1, we derive that there exists $g \in \operatorname{Ker}\Psi$ such that $e \in D(\mathscr{E}(f, g))$, or, to put it another way, there exists $g \in \operatorname{Ker}\Psi$ such that $D_g \subset D_f$. The second conclusion leads to

$$\sup\{\mathscr{E}(\mathrm{f}, \mathrm{g})(\mathrm{e}) \geqq 0 \colon g \in \operatorname{Ker}\Psi \text{ such that } D_g \subset D_f\}$$

which may also be written as

$$\sup\{(\sup\{s\colon s \leqq f(z)\text{-}g(z)\}\colon g \in \operatorname{Ker}\Psi \text{ such that } D_g \subset D_f\} \geqq 0.$$

From this we immediately conclude that there exists $g \in \operatorname{Ker}\Psi$ with $D_g \subset D_f$ and $g(z) \leqq f(z)$, $\forall z \in D_g$, which means that $f \in \operatorname{Ker}\Psi$; hence $\operatorname{Ker}\Psi_1 \subset \operatorname{Ker}\Psi$.

Conversely, suppose that $f \in \operatorname{Ker}\Psi$; then there exists $g \in \operatorname{Ker}\Psi$ such that $D_g \subset D_f$ and $g(z) \leqq f(z)$, $\forall z \in D_g$. In order for $f \in \operatorname{Ker}\Psi_1$ we have to show that $e \in D(\Psi_1(f))$ and $\Psi_1(f)(e) \geqq 0$.

Now, from Lemma 1, $e \in D(\Psi_1(f))$ iff there exists $g \in \operatorname{Ker}\Psi$ such that $e \in D(\mathscr{E}(f, g))$, which is true since $D_g \subset D_f$ for some $g \in \operatorname{Ker}\Psi$.

Again from Lemma 1, $\Psi_1(f)(e) = \sup\{\mathscr{E}(\mathrm{f},\mathrm{g})(\mathrm{e})\colon g \in \operatorname{Ker}\Psi \text{ such that } D_g \subset D_f\}$, with $\mathscr{E}(f, g)(e) = \sup\{s \in V\colon s \leqq f(z) - g(z)\}$, and this is already non-negative for one particular g; hence $\Psi_1(f)(e) \geqq 0$. So we also have $\operatorname{Ker}\Psi \subset \operatorname{Ker}\Psi_1$. According to Corollary 1, $\Psi = \Psi_1$, which proves the theorem.

We finally remark that, when taking in Theorem 1 $V = \{0\}$ and $G = (R^2, +)$, we obtain as a special case the Matheron representation theorem for subsets R^2.

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