10_{101} HAS NO PERIOD 7: A CRITERION FOR PERIODIC LINKS

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Abstract. We prove that the prime knot 10_{101} has no period 7 by analyzing its Jones polynomial.

The purpose of this paper is to introduce a convenient criterion for periodicity of links in \( S^3 \). Our tool will be the Kauffman bracket polynomial \( \langle \rangle \) taking values in \( Z[A^{-1}, A] \) and the normalized version \( f \) which is an ambient isotopy invariant with values in \( Z[A^{-2}, A^2] \) (see [K]). Let us define a reduction homomorphism from \( Z[A^{-1}, A] \) into the group ring over \( Z_p \) of the multiplicative cyclic group \( C_{p^n} \) by reducing the coefficients modulo \( p \) and putting \( A \rightarrow \gamma \), where \( \langle \gamma \rangle = C_{p^n} \). We will denote by \( f_{p,n} \) the image of \( f \) under this homomorphism. Then we have the following.

Theorem. If \( L \) has period \( p^n, n \geq 1 \) (\( p \) a prime), then \( f_{p,n}(L) \) is a symmetric element in \( Z_p[C_{p^n}] \), that is the coefficient of \( \alpha \in C_{p^n} \) is the same as that of \( \alpha^{-1} \).

Corollary. 10_{101} has no period 7: The \( f \) polynomial of 10_{101} is \( A^{-48} - A^{-44} + 7A^{-40} - 11A^{-36} + 13A^{-32} - 14A^{-28} + 14A^{-24} - 10A^{-20} + 7A^{-16} - 3A^{-12} + A^{-8} \), so the reduced version is \( 3A^{-2} + 4A^{-1} + 5A + 4A^2 + 6A^3 \) which is not symmetric.

Murasugi [M] has recently shown, that 10_{105}—one of the other undecided cases in Burde-Zieschang table (see [B-Z])—has no period 7. Our criterion also works in this case, the reduced polynomial being equal \( A^{-2} + 5A^{-1} + 4 + 6A + 3A^2 + 3A^3 \).

Proof of the theorem. We will work with a periodic diagram \( D \) of the considered periodic link. That is \( D \) is invariant under the rotation \( \rho \) of the projection plane around the center 0 disjoint from \( D \). This makes it possible to restrict our attention to the bracket polynomial rather than the \( f \) polynomial, because by definition \( f(L) = (-A)^{-3w(L)}\langle L \rangle \), and for a periodic diagram of an oriented link \( L \) the twist number \( w(L) \) is obviously divisible by \( p^n \), whence...
\((-A)^{-3m(L)} = \pm 1\) and \(f_{p,n}(L) = \pm \langle L \rangle_{p,n}^c\). Thus we only need to prove that the reduced bracket polynomial of a periodic diagram is symmetric.

In computing the bracket polynomial of \(D\) we consider the states of \(D\), which are obtained by splitting every crossing of the diagram in one of the two possible ways (a positive split \(\times \rightarrow \infty\), the NW-SE segment being an overpass, or a negative split \(\times \rightarrow \), the NW-SE segment being an overpass). Each of the states has its contribution in the bracket expansion: every split scores \(A\) or \(A^{-1}\), and the product of these coefficients multiplied by \((-A^{-2} - A^2)^{|S| - 1}\) is the contribution of \(S\) in the bracket polynomial of \(D\) (\(|S|\) denotes the number of components). Thus the contribution of \(S\) is a summand of the form \(A^k(-A^{-2} - A^2)^{|S| - 1}\). The group \(\langle \rho \rangle\) acts on the set of states of \(D\). Obviously the contributions of two states belonging to one orbit are equal. Thus the total contribution of a nontrivial orbit to the reduced bracket is trivial (because there are \(p^k\) equivalent states in such an orbit, \(k \geq 1\)).

Let us consider a periodic state \(S\). If this state is obtained by a positive split at a crossing \(c\), then the invariance of \(S\) implies that all the images of \(c\) under the action of \(\langle \rho \rangle\) are also positively split. Thus the splits at these crossings contribute \(A^{|n|}\) to the bracket of \(S\) and this is equal 1 in the reduced situation. It follows that the contribution of \(S\) to the reduced bracket is simply \(\langle S \rangle = (-A^{-2} - A^2)^{|S| - 1}\), and this is obviously symmetric. This completes the proof.

**References**


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