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FIXED POINTS OF UNITARY $\mathbf{Z}/p^s$-MANIFOLDS

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ABSTRACT. Let $G = \mathbf{Z}/p^s$ (p an odd prime). We show that restricting the local representations in a unitary G-manifold M with isolated fixed points results in severe restrictions on the number of fixed points (counted with the sign of their orientation), paralleling results obtained by Conner and Floyd in the case $G = \mathbf{Z}/p$. Specifically, the number of noncancelling fixed points is either zero or divisible by p^n, where $n \to \infty$ as the dimension of $M \to \infty$. This result also parallels phenomena in framed G-manifolds, as discussed by the first author in a previous paper.

INTRODUCTION

Conner and Floyd proved the following result in [CF, 40.1]. Let $G = \mathbf{Z}/p$ (p an odd prime) and let M be a smooth unitary n-dimensional G-manifold with isolated fixed points. Assume also that the local representations normal to the fixed points coincide. Then if we denote the collection of fixed points, counted with orientation, by Y, one has $[Y] \in p^{a(n)}\Omega_0^U$, where $\Omega_0^U \cong \mathbf{Z}$ is zero dimensional unitary bordism, and where $a(n) \to \infty$ as $n \to \infty$. In particular, there cannot exist any unitary G-manifold with a single fixed point.

Here we show that the Conner and Floyd result generalizes directly to the case $G = \mathbf{Z}/p^s$ for arbitrary s. (The precise result is stated in §2.) Note that the generalization follows easily by induction on the order of G if either the local representation possesses a nonzero fixed subspace by some nontrivial subgroup K or if the manifold M in question contains only isolated fixed points by G; in the first instance one can restrict to the fixed set by K, and in the second instance one can regard M as a K-manifold. The general case, in which M has isolated G-orbits of the form G/K for $K \neq G$, is far less tractable. The analogous result for framed G-manifold appears in fact to require the full force of the Segal conjecture [W1]. In the absence of an analogous result for unitary G-bordism, our proof here makes use of the eta invariant of Atiyah-Patodi-Singer [APS], and uses the combinatorial formulas of Gilkey [G1].

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2. Oriented G-manifolds modelled on a representation

In [W1] the geometry of G-manifolds modelled on a fixed virtual representation $\gamma = (V - W) \in RO(G)$ is studied. Such G-manifolds were originally considered by Pulikowski [P1] and Kosniowski [K1]. Here we recall the basic definitions.

The letters V, W, Y and Z will be used to denote finite dimensional G-invariant subspaces of the orthogonal G-module $\mathscr{U} = R^{\infty}$, where R denotes the regular representation of G endowed with its natural inner product. For brevity, we write $V < \mathscr{U}$, $W < \mathscr{U}$, and so on.

Definition 2.1. Let $\gamma = (V - W) \in RO(G)$ and let M be a compact smooth G-manifold. Then M has *equivariant dimension* γ if, for each $x \in \operatorname{Int} M$ (with isotropy subgroup $G_x \subset G$), there is a smooth G_x-equivariant imbedding onto an open set, $\theta_x \colon Y_x \to M$ taking 0 to x, where Y_x is a G_x-module such that $Y_x \oplus W \cong V$ as a G_x-module. Thus Y_x represents the element $\gamma|_{G_x} \in RO(G_x)$. We refer to such a manifold as a (smooth) *γ-manifold.* If γ is represented by an actual G-module $V < \mathscr{U}$, we refer to a γ-manifold as a *V-manifold.*

Examples of γ-manifolds include G-manifolds all of whose fixed sets are connected and nonempty, and the boundaries of such manifolds; if M is a γ-manifold, then ∂M is a $(\gamma - 1)$-manifold. The tangent bundle τ_M of a γ-manifold M is also γ-dimensional in the sense that the fiber over a typical point x is G_x-equivalent to Y_x, where Y_x is as above. Similarly, its normal bundle ξ_M is Z-dimensional for some (sufficiently large) G-module Z.

In the language of V-manifolds, our result takes the following form.

Theorem. *Let $G = \mathbf{Z}/p^s$, assume that $V < \mathscr{U}$ has $V^G = 0$, and let M be any unitary G-manifold of dimension nV $(= V \oplus \cdots \oplus V)$. Then the G-fixed set Y lies in $p^{r(n)}\Omega_0^U$, where $r(n) \to \infty$ as $n \to \infty$.*

In §3, we reduce the theorem to a statement about induction from the maximal proper subgroup (Proposition 3.2), and prove the reduction in §5.

Corollary. *No unitary G-manifold of dimension V can possess a single isolated point fixed by G.*

Proof. If the unitary G-manifold M contains a single isolated fixed point, then so does $M^n = M \times \cdots \times M$ for arbitrary $n > 0$. Since the latter has dimension nV, this is a contradiction. □

3. The eta invariant

Let $R(U)$ denote the complex representation ring of the unitary group, and let $R_0(G)$ be the augmentation ideal of the representation ring of G. Then Gilkey [G1] discusses the eta invariant for free G-manifolds

$$\eta \colon R_0(G) \otimes R(U) \otimes \widetilde{\Omega}_*^U(BG_+) \to \mathbf{Q}/\mathbf{Z}$$

which completely detects unitary free G-bordism when G is any spherical space form group. That is, if $\eta(\theta, M) = 0$ for all $\theta \in R_0(G) \otimes R(U)$, then $M = 0$. In addition, Gilkey establishes the following properties.

Lemma 3.1. (a) *If $H \subset G$ and if M is a free unitary H-manifold, then $\eta_G(\theta, G \times_H M) = \eta_H(\theta|_H, M)$, where η_K denotes the eta invariant with respect to the ambient group K, and where $\theta|_H = r \otimes 1(\theta)$, where $r\colon R_0(G) \to R_0(H)$ denotes the restriction homomorphism.*

(b) *If $H \subset G$ and if M is a free unitary G-manifold, then $\eta_H(\theta, M) = \eta_G(\imath(\theta), M)$, where $\imath(\theta) = (\mathrm{ind}_H^G \otimes 1)(\theta)$.*

(c) *If $\rho \in R_0(G)$ and W is any fixed-point free representation of G, then $\eta(\rho, S(W)) = \langle \rho, \beta(W)\rangle_G$, where the inner product is given on complex class functions by $\langle f_1, f_2\rangle = |G|^{-1}\sum_{g\in G} f_1(g)f_2(g)$, and where the construction of $\beta(W)$ is given as follows. Let τ denote the class function underlying W, and take $\alpha(\tau) = \det(\tau - 1)/\det(\tau)$. Then $\beta(W)(g) = \mathrm{Tr}(\alpha(\tau(g)))^{-1}$ if $g \neq 1$, and 0 otherwise.* □

Further, one sees [G1, remarks after Lemma 2.3] that if $\varphi \in R(U)$ and $\rho \in R_0(G)$, then $\eta(\rho \otimes \varphi, S(W)) = \eta(\rho \cdot \psi, S(W))$, where $\psi \in R(G)$ is suitably constructed.

Now let V be any one-dimensional free unitary representation of $G = \mathbf{Z}/p^s$ ($s \geq 2$), and let $K = \mathbf{Z}/p^{s-1} \subset G$. Denote the unit sphere in V^n by SnV.

Proposition 3.2. *Let $[S_n] = p[SnV] - [G \times_K SnV]$. Then there exists $\rho \in R_0(G)$ such that $\eta_G(\rho, S_n)$ has order $p^{r(n)}$, where $r(n) \to \infty$ as $n \to \infty$.*

The proof of Proposition 3.2 occupies §§4, 5. We should remark that the exponent $r(n)$ behaves erratically as a function of p and s.

Proof of theorem. Assume the theorem holds for $G = \mathbf{Z}/p^r$ with $r \leq s-1$, and let M be a unitary G-manifold of dimension nV and fixed set Y. Assume first that $V^K \neq 0$ for some nontrivial subgroup $K \subset G$. Then M^K is a G/K-manifold of dimension nV^K with G/K-fixed set Y, and the conclusion follows by the induction hypothesis. If on the other hand V is a free representation (so that $S(V)$ is free), consider the nV-dimensional unitary G-manifold

$$N = pM - G \times_K M,$$

where $K = \mathbf{Z}/p^{s-1}$ (and the minus sign indicates oppositely oriented complex structure on a trivial two-dimensional summand of the normal bundle). If M possesses a G-orbit Q of isolated fixed points with isotropy subgroup J of K, then $G \times_K M$ possesses p copies of $-Q$, so that the corresponding fixed point orbits cancel in N. By attaching G-handles, one may remove such orbits to obtain a unitary nV-dimensional G-manifold N with isolated fixed points a copy C of $[Y](p(G/G) - G/K)$, corresponding to the G-fixed set Y of M. The complement of a regular neighborhood of C now gives a free null-bordism of $[Y][S_n]$, and by Proposition 3.2 $[Y]$ has the form $p^{r(n)}$, where $r(n) \to \infty$ as $n \to \infty$. $\square$

4. Divisibility properties in $R_0(G)$

Let $G = \mathbf{Z}/p^s$, and let $\widetilde{R}(G)$ be the $R(G)$-module $R(G)/\rho R(G)$, where ρ is the regular representation of G. Let $\widetilde{R}_0(G)$ be the image of $R_0(G)$ in $\widetilde{R}(G)$ induced by the inclusion. The eta invariant then factors through $\widetilde{R}_0(G)\otimes R(U)\otimes \widetilde{\Omega}(B\mathbf{Z}/p^s_+)$, since $\widetilde{\Omega}(B\mathbf{Z}/p^s_+)$ is generated as a Ω_*-module by the $[SnV]$, where the determining formula [G1, Lemma 2.3] ignores the values of the relevant class functions at $1 \in G$. We consider divisibility in $\widetilde{R}(G)$ by $(1-\omega)$, where ω is the character of the canonical one-dimensional unitary representation of G.

Let $\varepsilon\colon \mathbf{Z}[x] \to \mathbf{Z}$ be the augmentation, and let $\mathscr{T} = \varepsilon^{-1}(p^s\mathbf{Z})$. Let $\theta\colon \mathscr{T} \to \widetilde{R}(G)$ be given by evaluation at ω. Then $\operatorname{Im}\theta \subset \widetilde{R}_0(G)$, since, for $p \in \mathscr{T}$, writing $p(x) = \sum_{i=1}^{kp^s} \lambda_i x^{n(i)}$ with $\lambda_i = \pm 1$, $\sum_i \lambda_i \equiv 0 \bmod p^s$, one observes that $\sum_i \lambda_i \omega^{n(i)} \approx \sum_i \lambda_i \omega^{n(i)} + \sum_{\lambda_i=-1} \rho - kp^s \sum_{\lambda_i=-1} \rho$ for suitable k, which is divisible by $(1-\omega)$.

Lemma 4.1. *In $\widetilde{R}(G)$, one has, for any $m > 0$, a sequence $(t_i \geq 0)$ with $t_1 > 0$ and $\sum_{i=1}^n t_n = m$, and a sequence $(a_i \in \mathbf{Z} - p^s\mathbf{Z})$ such that*

$$(*) \qquad \theta(p(x)) = (1-\omega)^{t_1}(a_1 + (1-\omega)^{t_2}(a_2 + \cdots + (1-\omega)^{t_n} q_n)\ldots)$$

for some $q_n \in \widetilde{R}_0(G)$. Further, if $\theta(p(x))$ restricts to 0 in $\widetilde{R}(H)$, where $H = \mathbf{Z}/p^{s-1} \subset G$, then $a_1 \in p^{s-1}\mathbf{Z}$.

Proof. The first assertion follows easily by the above remarks, since one may ensure divisibility by $(1-\omega)$ at any stage by adding suitable $a_i \in \mathbf{Z} - p^s\mathbf{Z}$ in order to land in $\operatorname{Im}\theta$. For the second assertion, passing to $\mathbf{Z}/p^{s-1}$, $p(\omega)$ restricts to $N\rho$ as an element of $R(H)$ where ρ is the regular representation of H. Applying the augmentation map to both sides now gives $N = 0$, so that $p(\omega) = 0$ in $R(H)$. But if $(1-\omega)x = 0$ in $R(H)$, then $x = k\rho$, so that we now have

$$(a_1 + (1-\omega)^{r_1}(a_2 + (1-\omega)^{r_2}(\cdots)\cdots) = 0$$

in $\widetilde{R}(H)$. Thus equals $N\rho$ in $R(H)$ again, so $a_1 \equiv 0 \mod p^{s-1}$. $\square$

This factorization is far from unique; one may replace $\theta(p(x))$ by $\theta(p(x)) + \omega^i(\omega^{p^s} - 1)$, where $\omega^i(\omega^{p^s} - 1)$ may be expressed as a polynomial in $(\omega - 1)$ with integral coefficients. Further, if we relax the requirement that $a_i \notin p^s\mathbf{Z}$, then adding a suitable multiple of $\omega^{p^s} - 1$, whose coefficient of $(1-\omega)^{p^s}$ is 1, yields an expansion of the form $(*)$ with at least one coefficient prime to p.

We shall take

$$\mathfrak{p}(x) = 1 + x + \cdots + x^{p^{s-1}-1} + 1 + x^{p^{s-1}+1} + \cdots + x^{2p^{s-1}-1} + 1 + x^{2p^{s-1}+1} + \cdots + x^{(p-1)p^{s-1}-1} + 1 + x^{(p-1)p^{s-1}+1} + \cdots + x^{p^s-1},$$

and use an expansion of the form $(*)$ with the coefficient of $(1-\omega)^{p^s}$ prime to p, and where the expansion will be continued as far as necessary.

5. Consequences in free $\mathbf{Z}/p^s$-bordism

Before proving the theorem, we need a preliminary result on free $\mathbf{Z}/p^s$-bordism. Let $a(n)$ denote the exponent in the order of $[SnV]$ in free $\mathbf{Z}/p$ bordism.

Now consider the order of

$$S_n = p[SnV] - [G \times_H SnV],$$

where $H = \mathbf{Z}/p^{s-1} \subset \mathbf{Z}/p^s$ and V has character ω^{-1}. Since $[G \times_H SnV] = \imath[SnV|_H]$, one has, for $\sigma \in R_0(G)$,

$$\begin{aligned}
\eta(\sigma, S_n) &= \left\langle p\sigma, \frac{\omega^{-n}}{(1-\omega^{-1})^n} \right\rangle - \left\langle \imath(\sigma), \frac{\omega^{-n}}{(1-\omega^{-1})^n} \right\rangle \\
&= \langle (p-\imath)(1)\cdot\sigma, (1-\omega)^{-n}\rangle = \langle \mathfrak{p}(\omega)\cdot\sigma, (1-\omega)^{-n}\rangle \\
&= \langle (1-\omega)^{t_1}(a_1 + (1-\omega)^{t_2}(a_2 + \cdots + (1-\omega)^{t_n} q_n)\cdots)\cdot\sigma, (1-\omega)^{-n}\rangle \\
&= \sum_{i\geq 1} \langle a_i \cdot \sigma, (1-\omega)^{t_1+\cdots+t_i-n}\rangle, \\
&= \sum_{i\geq 1} a_i \eta(\sigma, S(n-u_i)V),
\end{aligned}$$

where $u_i = \sum_{j=1}^{i} t_j$, and where the sum on the left terminates when $u_{i+1} > n$,

$$= \eta\left(\sigma, \sum_{i\geq 1} a_i S(n-u_i)V\right).$$

Note that the a_i are independent of n (except insofar as the numbers of terms we use), and that at least one a_i may be taken to be prime to p. Partition the set of integers ≥ 1 as $\mathscr{P}_1 \coprod \mathscr{P}_2$, where $k \in \mathscr{P}_1$ iff $a_k \equiv 0 \mod p$. Then one has, with $b_i = a_i/p$ for $i \in \mathscr{P}_1$,

$$\begin{aligned}
\eta\left(\sigma, \sum_{i\geq 1} a_i S(n-u_i)V\right) &= \sum_{i\in\mathscr{P}_1} b_i \eta(\sigma, pS(n-u_i)V) + \sum_{k\in\mathscr{P}_2} a_k \eta(\sigma, S(n-u_k)V) \\
&= \sum_{i\in\mathscr{P}_1} b_i [\eta_H(\sigma, S(n-u_i)V) + \eta(\sigma, S_{n-u_i})] + \sum_{k\in\mathscr{P}_2} a_k \eta(\sigma, S(n-u_k)V) \\
&= \sum_{i\in\mathscr{P}_1} b_i \left[\eta_H(\sigma, S(n-u_i)V) + \eta\left(\sigma, \sum_{j\geq 0} a_j S(n-u_i-u_j)V\right)\right] \\
&\quad + \sum_{k\in\mathscr{P}_2} a_k \eta(\sigma, S(n-u_k)V) \\
&= \sum_{i\in\mathscr{P}_1} b_i \eta_H(\sigma, S(n-u_i)V) + \sum_{j\in\mathscr{P}_1, i\geq 0} a_i b_j \eta(\sigma, S(n-u_i-u_j)V) \\
&\quad + \sum_{k\in\mathscr{P}_2} a_k \eta(\sigma, S(n-u_k)V) \\
&\quad \vdots \\
&= \sum_{i\in\mathscr{P}_1} b_i \eta_H(\sigma, S(n-u_i)V) + \sum_{i,j\in\mathscr{P}_1} b_i b_j \eta_H(\sigma, S(n-u_i-u_j)V) + \cdots \\
&\quad + \sum_{k\in\mathscr{P}_2} a_k \eta(\sigma, S(n-u_k)V) + \sum_{i\in\mathscr{P}_2, j\in\mathscr{P}_1} a_i b_j \eta(\sigma, S(n-u_i-u_j)V) + \cdots .
\end{aligned}$$

Multiplying through by p now gives

$$\begin{aligned}
&\sum_{i\in\mathscr{P}_1} a_i \eta_H(\sigma, S(n-u_i)V) + \sum_{i,j\in\mathscr{P}_1} a_i b_j \eta_H(\sigma, S(n-u_i-u_j)V) + \cdots \\
&\quad + \sum_{k\in\mathscr{P}_2} a_k \eta(\sigma, pS(n-u_k)V) + \sum_{i\in\mathscr{P}_2, j\in\mathscr{P}_1} a_i b_j \eta(\sigma, pS(n-u_i-u_j)V) + \cdots \\
&= \sum_{i\geq 1} a_i \eta_H(\sigma, S(n-u_i)V) + \sum_{i\geq 1, j\in\mathscr{P}_1} a_i b_j \eta_H(\sigma, S(n-u_i-u_j)V) + \cdots \\
&\quad - \left(\sum_{k\in\mathscr{P}_2} a_k \eta_H(\sigma, S(n-u_k)V) + \sum_{i\in\mathscr{P}_2, j\in\mathscr{P}_1} a_i b_j \eta_H(\sigma, S(n-u_i-u_j)V)\right) \\
&\quad + \sum_{k\in\mathscr{P}_2} a_k \eta(\sigma, pS(n-u_k)V) + \sum_{i\in\mathscr{P}_2, j\in\mathscr{P}_1} a_i b_j \eta(\sigma, pS(n-u_i-u_j)V) + \cdots \\
&= \sum_{i\geq 1} a_i \eta_H(\sigma, S(n-u_i)V) + \sum_{i\geq 1, j\in\mathscr{P}_1} a_i b_j \eta_H(\sigma, S(n-u_i-u_j)V) + \cdots \\
&\quad + \sum_{k\in\mathscr{P}_2} b_k \eta(\sigma, S_{n-u_k}) + \sum_{i\in\mathscr{P}_2, j\in\mathscr{P}_1} a_i b_j \eta(\sigma, S_{n-u_i-u_j} V) + \cdots .
\end{aligned}$$

The first batch of summands now vanishes, leaving

$$p\eta(\sigma, S_n) = a_{k_0}\eta(\sigma, S_{n-u_{k_0}}V) + \text{terms in lower dimensional spheres of the form } c_k\eta(\sigma, S_{n-r_k}V),$$

where k_0 is the smallest element in $\mathscr{P}_2$, and where a_{k_0} is prime to p. If there exists an n such that $S_n \neq 0$, then an easy induction using the Smith homomorphism and appropriate choices for n shows that the exponent of p in the order of S_n approaches ∞ as $n \to \infty$, completing the proof. But $S_n \neq 0$ since the order of SV is p^s, whereas the coefficient of $(1-\omega)^{u_1}$ in the *original* (Lemma 4.4) expansion of $\theta(p(x))$ is not in $p^s\mathbf{Z}$, and we can arrange that $n - u_1 = 1$ there. □

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