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PRODUCTS OF INFINITE-DIMENSIONAL SPACES

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ABSTRACT. Observations concerning the product of R. Pol's weakly infinite-dimensional uncountable-dimensional compactum with various spaces are made. A proof showing that the product of a C-space and a compact C-space is again a C-space is given. Related questions, motivated by this result, are asked.

1. INTRODUCTION

By a space we mean a metric space and by the dimension of a space we mean the Lebesgue covering dimension, for example as presented in [E1]. A space is called *countable-dimensional* if it can be written as the union of a countable number of finite-dimensional subspaces. If these subspaces can be chosen to be closed in X, then X is said to be *strongly countable-dimensional.* A space X is *weakly infinite-dimensional* (in the sense of Alexandroff) if for every countable family of pairs of disjoint closed subsets $\{(A_n, B_n): n \in \mathbf{N}\}$ there exists a closed separator $S_n \subset X$ of each pair (A_n, B_n) so that the $\bigcap\{S_n: n \in \mathbf{N}\} = \varnothing$. If a space X is not weakly infinite-dimensional, then the space is said to be *strongly infinite-dimensional.* An excellent survey of these topics is contained in [EP].

A space X has *property* C and is said to be a C*-space* if for any given sequence of open covers $\{\mathscr{U}_n: n \in \mathbf{N}\}$ there exists a refinement $\mathscr{V}_n$ of $\mathscr{U}_n$, for each $n \in \mathbf{N}$, so that each $\mathscr{V}_n$ is a collection of pairwise disjoint open subsets of X; such a refinement is called a C*-refinement*, with the $\bigcup\{\mathscr{V}_n: n \in \mathbf{N}\}$ forming a cover of X. This covering property was originally defined by Haver for metric spaces and later generalized by Addis and Gresham to more general topological spaces. Every countable-dimensional space has property C, and every space with property C is weakly infinite-dimensional [H] [AG]. In particular, every finite-dimensional space has property C.

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The following are well-known facts following directly from the definitions. More detail may be found in [G] and [AG].

Subspace Theorems. *Every closed subspace of a weakly infinite-dimensional space (a C-space) is weakly infinite-dimensional (a C-space).*

Sum Theorems. *If a space $X = \bigcup\{X_n : n \in \mathbf{N}\}$, where each subspace X_n is weakly infinite-dimensional (a C-space), then X is also weakly infinite-dimensional (a C-space).*

Local Theorems. *If every point of a space X has a neighborhood which is weakly infinite-dimensional (a C-space), then X is also weakly infinite-dimensional (a C-space).*

It is not known whether every weakly infinite-dimensional space has property C; however, R. Pol has constructed a weakly infinite-dimensional compactum containing a strongly infinite-dimensional subspace. Hence, this compactum cannot be countable-dimensional [P1]. When constructed as a subspace of the Hilbert cube, Pol's compactum P, which is known to have property C, has the form $P = X \cup Z$ where X is a topologically complete totally disconnected strongly infinite-dimensional subspace of the Hilbert cube with $Z = P \backslash X$ countable-dimensional. Thus, there are disjoint Bernstein sets B_1 and B_2, i.e., all compact subsets of B_1 or B_2 are countable, so that $Z = B_1 \cup B_2$ [P2].

2. Examples

It is obvious from the subspace theorems that if X and Y are spaces whose product $X \times Y$ is weakly infinite-dimensional (a C-space), then both factors X and Y must also be weakly infinite-dimensional (C-spaces).

The general converse to this statement is false. Indeed, using the notation given at the end of §1, it is known that $B_1 \cup X$ and $X \cup B_2$ have property C [EP], and thus are weakly infinite-dimensional. However, the product $(B_1 \cup X) \times (X \cup B_2)$ contains $X \times X$ as a closed subspace, and so must be strongly infinite-dimensional [P2]. Neither factor could be compact, for if say $B_1 \cup X$ were compact, then the projection to the other factor $X \cup B_2$ would be a closed mapping. This would make X a closed strongly infinite-dimensional subspace of $X \cup B_2$, contradicting the weak infinite-dimensionality of $X \cup B_2$. This example motivates the following question which we discuss in §3 of this paper.

Question 1. When is the product of two weakly infinite-dimensional spaces (C-spaces) again weakly infinite-dimensional (a C-space)?

It seems difficult to formulate positive product theorems for these infinite-dimensional dimension theories. Addis and Gresham have given the following product theorem for property C [AG].

Theorem 1. *Let X and Y be two C-spaces. If Y is compact and has a basis $\mathscr{B}$ of open sets such that for all $B \in \mathscr{B}$ the product $X \times Bdy(B)$ has property C, then the product $X \times Y$ also has property C.*

Corollary. *If a space X has property C and Y is a σ-compact strongly countable-dimensional space, then $X \times Y$ has property C.*

Proof. Since the space Y is σ-compact, $Y = \bigcup\{Y_n : n \in \mathbf{N}\}$ where each Y_n is a compact. Moreover, since Y is a strongly countable-dimensional space it is possible to write $Y = \bigcup\{Z_m : m \in \mathbf{N}\}$ where each Z_m is a closed finite-dimensional subspace of Y. Since $Y_n = \bigcup\{Z_m \cap Y_n : m \in \mathbf{N}\}$ for each $n \in \mathbf{N}$, it is seen that each compact subspace Y_n is also strongly countable-dimensional. An inductive application of the theorem gives that each produce $X \times (Z_m \cap Y_n)$ has property C, and thus we can apply the sum theorem once to $X \times Y_n = \bigcup\{X \times (Z_m \cap Y_n) : m \in \mathbf{N}\}$ and again to $X \times Y = \bigcup\{X \times Y_n : n \in \mathbf{N}\}$ to see that $X \times Y$ has property C. □

Y. Hattori has pointed out that Y needs only to be a σ-compact countable-dimensional space, since a compact countable-dimensional space has small transfinite inductive dimension [E2, Corollary 4.7]. A simple argument using transfinite induction will then give the result. However, E. Pol has shown that the compactness of the countable-dimensional factor cannot be dropped [P].

Theorem 2. *There exists a C-space whose product with a subspace of the irrationals is strongly infinite-dimensional, and thus cannot have property C.*

3. Product theorems

In this section, we consider products of C-spaces, and, in particular, we consider the product of Pol's compactum with various weakly infinite-dimensional spaces.

Theorems 3. *The product of a C-space with a compact C-space is again a C-space.*

Proof. Let X and Y be C-spaces with Y compact. Given a sequence of open covers of the product $X \times Y$, we rewrite the sequence as a countable collection $\{\{\mathscr{U}_{m,n} : n \in \mathbf{N}\} : m \in \mathbf{N}\}$ of sequences of open covers. Moreover, we may assume that each cover $\mathscr{U}_{m,n}$ is of the form

$$\mathscr{U}_{m,n} = \{A^{\alpha}_{m,n} \times B^{\alpha}_{m,n} : \alpha \in \Gamma_{m,n}\},$$

where each $A^{\alpha}_{m,n}$ is open in X and every $B^{\alpha}_{m,n}$ is open in Y.

Fix $m \in \mathbf{N}$ and let $x \in X$ be fixed but arbitrary. For each $n \in \mathbf{N}$, we use the compactness of Y to choose a finite subset $\Gamma_{m,n}(x)$ from the indexing set $\Gamma_{m,n}$ so that

$$\mathscr{B}_{m,n}(x) = \{B^{\alpha}_{m,n} : \alpha \in \Gamma_{m,n}(x)\}$$

is a finite cover of Y with $x \in A^{\alpha}_{m,n}$ for each $\alpha \in \Gamma_{m,n}(x)$. Because Y has property C, for each $n \in \mathbf{N}$ we can choose a C-refinement $\mathscr{D}_{m,n}(x)$ of $\mathscr{B}_{m,n}(x)$ so that the $\bigcup\{\mathscr{D}_{m,n}(x) : n \in \mathbf{N}\}$ is a cover of Y.

We use the compactness of Y again, this time to choose a positive integer $r_m(x) \in \mathbf{N}$ so that the $\bigcup\{\mathscr{D}_{m,n}(x)\colon n = 1, \ldots, r_m(x)\}$ is a finite subcover of Y, and then we set

$$A_m(x) = \bigcap\{A^{\alpha}_{m,n}\colon n = 1, \ldots, r_m(x), \alpha \in \Gamma_{m,n}(x)\}.$$

Since $\Gamma_{m,n}(x)$ is a finite set, $A_m(x)$ is an open neighborhood of x in X. Thus, by constructing such a $A_m(x)$ for each $x \in X$ and defining

$$\mathscr{A}_m = \{A_m(x)\colon x \in X\},$$

we obtain an open cover of X.

In this manner, for each $m \in \mathbf{N}$ we construct such an open cover $\mathscr{A}_m$ of X. Since X has property C, we can choose a C-refinement $\mathscr{C}_m$ of each $\mathscr{A}_m$ such that the $\bigcup\{\mathscr{C}_m\colon m \in \mathbf{N}\}$ covers X. Since each $\mathscr{C}_m$ is a refinement of $\mathscr{A}_m$, we can choose a function $\phi_m\colon \mathscr{C}_m \to X$ so that for each $C \in \mathscr{C}_m$ we have

$$C \subset A_m(\phi_m(C)),$$

and thus for any $n \in \{1, \ldots, r_m(\phi_m(C))\}$ and $\alpha \in \Gamma_{m,n}(\phi_m(C))$ we have

$$C \subset A^{\alpha}_{m,n}.$$

Finally, for each fixed $m, n \in \mathbf{N}$, we define

$$\mathscr{V}_{m,n} = \{C \times D\colon n \in \{1, \ldots, r_m(\phi_m(C))\} \text{ and } D \in \mathscr{D}_{m,n}(\phi_m(C)) \text{ for some } C \in \mathscr{C}_m\}.$$

If $C \times D \in \mathscr{V}_{m,n}$, then $C \in \mathscr{C}_m$ with $D \in \mathscr{D}_{m,n}(\phi_m(C))$, and thus $D \subset B^{\alpha}_{m,n}$ for some $\alpha \in \Gamma_{m,n}(\phi_m(C))$. Therefore, we see that

$$C \times D \subset A^{\alpha}_{m,n} \times B^{\alpha}_{m,n} \in \mathscr{U}_{m,n},$$

so that $\mathscr{V}_{m,n}$ is an open refinement of $\mathscr{U}_{m,n}$. Moreover, since the elements of $\mathscr{C}_m$ are pairwise disjoint, and since for any fixed $C \in \mathscr{C}_m$ the elements of $\mathscr{D}_{m,n}(\phi_m(C))$ are pairwise disjoint, we see that the elements of $\mathscr{V}_{m,n}$ are also pairwise disjoint.

To show that $X \times Y$ has property C, it only remains for us to show that $X \times Y$ is covered by the $\bigcup\{\mathscr{V}_{m,n}\colon m, n \in \mathbf{N}\}$. Let (x, y) be an arbitrary point of $X \times Y$. Since the $\bigcup\{\mathscr{C}_m\colon m \in \mathbf{N}\}$ covers X, we can find $m \in \mathbf{N}$ and $C \in \mathscr{C}_m$ with $x \in C$. Since Y is covered by the $\bigcup\{\mathscr{D}_{m,n}(\phi_m(C))\colon n = 1, \ldots, r_m(\phi_m(C))\}$, we can find $n \in \{1, \ldots, r_m(\phi_m(C))\}$ and $D \in \mathscr{D}_{m,n}(\phi_m(C))$ so that $y \in D$. Hence,

$$(x, y) \in C \times D \in \mathscr{V}_{m,n},$$

which completes the proof. □

Corollary. *The product of Pol's compactum with any C-space has property C, and thus is weakly infinite-dimensional.*

Corollary. *For every* $n \in \mathbf{N}$, *the* n*-fold product* P^n *of R. Pol's compactum has property* C.

Corollary. *The product of a* σ*-compact* (*locally compact*) C*-space with a* C*-space is again a* C*-space.*

This last corollary follows from the sum and local theorems for property C.

Question 2. Let $f: X \to Y$ be an open and closed mapping between spaces X and Y, where Y is a C-space. If $f^{-1}(y)$ is a C-space for each $y \in Y$, then must X also be a C-space?

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