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UNIMODAL POLYNOMIALS ARISING FROM SYMMETRIC FUNCTIONS

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ABSTRACT. We present a general result that, using the theory of symmetric functions, produces several new classes of symmetric unimodal polynomials. The result has applications to enumerative combinatorics including the proof of a conjecture by R. Stanley.

1. INTRODUCTION

A sequence of real numbers $a_0, a_1, \dots, a_d$ is said to be *symmetric* if $a_i = a_{d-i}$ for $i = 0, \dots, \lfloor \frac{d}{2} \rfloor$ and is said to be *unimodal* if there exists an index $0 \le j \le d$ such that $a_1 \le a_2 \le \cdots \le a_{j-1} \le a_j \ge a_{j+1} \ge \cdots \ge a_d$. A polynomial $\sum_{i=0}^{d} a_i x^i \in \mathbf{R}[x]$ is called *symmetric* (respectively, *unimodal*) if the sequence $\{a_0, \dots, a_d\}$ has the corresponding property.

Unimodal polynomials arise often in combinatorics, geometry, and algebra and have been the subject of considerable research in recent years. Many different fields of mathematics have been used to prove that certain families of polynomials are unimodal such as, for example, classical analysis (see, e.g., [14, 22]), linear algebra (see, e.g., [10, 11]), the representation theory of Lie algebras and superalgebras (see, e.g., [12, 16, 17]), algebraic geometry (see, e.g., [19, 20]), the theory of total positivity (see, e.g., [2, 3]), the theory of symmetric functions (see, e.g., [4]), as well as the use of bijections and injections (see, e.g., [13]). We refer the reader to [20] for an excellent survey of many of these techniques and further references.

In this paper we also use the theory of symmetric functions to prove a general result (Theorem 2, below) that produces many new infinite families of symmetric unimodal polynomials. The result has a wide applicability, in particular, a conjecture and a result of R. Stanley both follow as specializations of it.

For brevity reasons, we will call a nonzero polynomial a Λ-*polynomial* if it has nonnegative coefficients and is both symmetric and unimodal. Note that we require a Λ-polynomial to be nonzero. If $p(x)$ is a Λ-polynomial then

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there is a unique $n \in \mathbf{N}$ such that $x^n p(1/x) = p(x)$. We call the number $n/2$ the *center of symmetry* of $p(x)$, and we write $C(p) = n/2$. So, for example, $C(x^2 + 3x^3 + x^4) = 3$ and $C(1 + x) = 1/2$. An elementary, though crucial property of Λ-polynomials, which will be used repeatedly in this paper, is the following.

Theorem 1. *Let $p(x)$ and $q(x)$ be two Λ-polynomials. Then $p(x)q(x)$ is a Λ-polynomial and $C(pq) = C(p) + C(q)$.*

Theorem 1 is well known and a proof of it can be found, e.g., in [20], Proposition 1.2, or in [1].

2. The main result

Let $x_1, x_2, x_3, \ldots$ be independent variables, and let $R[[x_1, x_2, \ldots]]$ be the ring of formal power series in $x_1, x_2, x_3, \ldots$ with coefficients in R, where R is a commutative ring with identity (we refer the reader to [8] for the definition and the basic properties of $R[[x_1, x_2, \ldots]]$). An element $p \in R[[x_1, x_2, \ldots]]$ is called *symmetric* if $p(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = p(x_1, x_2, \ldots)$ for all bijections $\sigma: \mathbf{P} \to \mathbf{P}$ (where $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$), and is said to be *bounded* if there is a constant M such that all the monomials appearing in p have degree $\leq M$. We let

$$\hat{\Lambda}_R \stackrel{\text{def}}{=} \{p \in R[[x_1, x_2, \ldots]]; p \text{ is symmetric}\}$$

and

$$\Lambda_R \stackrel{\text{def}}{=} \{p \in \hat{\Lambda}_R, p \text{ is bounded}\};$$

then Λ_R and $\hat{\Lambda}_R$ are subrings of $R[[x_1, x_2, \ldots]]$ and $\Lambda_R \subset \hat{\Lambda}_R$. We call $\hat{\Lambda}_R$ the ring of *symmetric formal power series* and Λ_R the ring of *symmetric functions*, with coefficients in R. Note that $\Lambda_R \subsetneq \hat{\Lambda}_R$ since, for example, $\prod_{i \geq 1}(1 + x_i) \in \hat{\Lambda}_R \backslash \Lambda_R$.

We will follow Chapter I of [9] for symmetric function notation and terminology. In particular, we will denote by s_λ (respectively, h_λ, e_λ, m_λ, and p_λ) the *Schur* (respectively, *complete homogeneous, elementary, monomial*, and *power sum*) symmetric functions, associated to the partition λ. We also denote by $\mathscr{P}$ the set all partitions, and by $\Lambda_{\mathbf{Q}}^n$ the subspace of all elements of $\Lambda_{\mathbf{Q}}$ that are homogeneous of degree n. If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0) \in \mathscr{P}$ then let $|\lambda| \stackrel{\text{def}}{=} \sum_{i=1}^{l} \lambda_i$, and also write $\lambda \vdash n$ if $n = |\lambda|$. We call a basis $\{a_\lambda\}_{\lambda \in \mathscr{P}}$ of $\Lambda_{\mathbf{Q}}$ *standard* if, for all $n \in \mathbf{N}$, $\{a_\lambda : \lambda \vdash n\}$ is a basis of $\Lambda_{\mathbf{Q}}^n$. Note that a standard basis $\{a_\lambda\}_{\lambda \in \mathscr{P}}$ is homogeneous and that $\deg(a_\lambda) = |\lambda|$.

We now come to the crucial definition of this paper. We say that an ordered triple $(\{a_\lambda\}_{\lambda \in \mathscr{P}}, \{b_\lambda\}_{\lambda \in \mathscr{P}}, \{c_\lambda\}_{\lambda \in \mathscr{P}})$ of standard bases of $\Lambda_{\mathbf{Q}}$ is *compatible* if:

(i) if $c_\lambda = \sum_\mu f_{\lambda\mu} a_\mu$ then $f_{\lambda\mu} \geq 0$, for all λ, $\mu \in \mathscr{P}$;

(ii) if $a_\lambda b_\mu = \sum_\nu g_{\lambda\mu}^\nu a_\nu$ then $g_{\lambda\mu}^\nu \geq 0$, for all λ, μ, $\nu \in \mathscr{P}$.

Conditions (i) and (ii) above are very mild and often automatically satisfied. In fact, we have the following result.

Proposition 1. *The ordered triple* $(\{a_\lambda\}_{\lambda\in\mathscr{P}}, \{b_\lambda\}_{\lambda\in\mathscr{P}}, \{c_\lambda\}_{\lambda\in\mathscr{P}})$ *is compatible in each of the following cases*:

(i) $a_\lambda = b_\lambda = c_\lambda \in \{m_\lambda, e_\lambda, h_\lambda, s_\lambda, p_\lambda\}$;
(ii) $a_\lambda = m_\lambda$, *and* $b_\lambda = c_\lambda \in \{m_\lambda, e_\lambda, h_\lambda, s_\lambda, p_\lambda\}$;
(iii) $a_\lambda = p_\lambda$, *and* b_λ, $c_\lambda \in \{p_\lambda, h_\lambda\}$;
(iv) $a_\lambda = s_\lambda$, *and* b_λ, $c_\lambda \in \{e_\lambda, h_\lambda, s_\lambda\}$.

The proof of the preceding Proposition follows immediately from the results in Chapter 1, §6 of [9] and from our definitions and is therefore omitted.

We are now in a position to state and prove the main result of this paper.

Theorem 2. *Let* $\{r_\mu(q)\}_{\mu\in S}$, $\{u_\nu(q)\}_{\nu\in T}$ *be two families of* Λ*-polynomials, where* S, $T \subseteq \mathscr{P}$ *and* $(0) \notin S$. *Suppose that there exists* n, $k \in \mathbf{Z}$ *such that*

$$C(r_\mu) = \frac{n|\mu|}{2}, \qquad C(u_\nu) = \frac{n|\nu| + k}{2},$$

for all $\mu \in S$, $\nu \in T$. *For* $\rho \in \mathscr{P}$, *define a polynomial* $R_\rho(q)$ *by*

$$\sum_{\rho\in\mathscr{P}} R_\rho(q) a_\rho = \frac{\sum_{\nu\in T} u_\nu(q) c_\nu}{1 - \sum_{\mu\in S} r_\mu(q) b_\mu}, \tag{1}$$

where $(\{a_\lambda\}_{\lambda\in\mathscr{P}}, \{b_\lambda\}_{\lambda\in\mathscr{P}}, \{c_\lambda\}_{\lambda\in\mathscr{P}})$ *is a compatible triple of standard bases of* $\Lambda_{\mathbf{Q}}$. *Then* $R_\rho(q)$ *is a* Λ*-polynomial and* $C(R_\rho) = (n|\rho| + k)/2$.

Proof. Note first that the symmetric formal power series $1 - \sum_{\mu\in S} r_\mu(q) b_\mu$ is an invertible element of $\mathbf{Q}[q][[x_1, x_2, \dots]]$ and that, therefore, the expression on the RHS of (1) is a well defined element of $\hat{\Lambda}_{\mathbf{Q}[q]}$. Now let

$$c_\nu = \sum_{\lambda\in\mathscr{P}} f_{\nu\lambda} a_\lambda \tag{2}$$

and

$$a_\rho b_\mu = \sum_\lambda g^\lambda_{\rho\mu} a_\lambda. \tag{3}$$

From our hypotheses and definitions it then follows that $f_{\nu\lambda} = 0$ unless $|\nu| = |\lambda|$, $g^\lambda_{\rho\mu} = 0$ unless $|\lambda| = |\rho| + |\mu|$, and

$$f_{\nu\lambda} \geq 0, \qquad g^\lambda_{\rho\mu} \geq 0$$

for all ν, λ, ρ, $\mu \in \mathscr{P}$. Now rewrite (1) as

$$\sum_{\rho\in\mathscr{P}} R_\rho(q) a_\rho - \sum_{\rho\in\mathscr{P}} \sum_{\mu\in S} R_\rho(q) r_\mu(q) a_\rho b_\mu = \sum_{\nu\in T} u_\nu(q) c_\nu. \tag{4}$$

Substituting (2) and (3) in (4) and then equating the coefficients of a_λ on both sides obtains

$$(5)\qquad \begin{cases} R_\lambda(q) = \sum_{\rho\in\mathscr{P}}\sum_{\mu\in S} R_\rho(q) r_\mu(q) g^\lambda_{\rho\mu} + \sum_{\nu\in T} u_\nu(q) f_{\nu\lambda} & \text{if } \lambda \neq 0, \\ R_{(0)}(q) = f_{(0)(0)} u_{(0)}(q). \end{cases}$$

Proceeding by induction on $|\lambda|$, the thesis is clearly true if $|\lambda| = 0$. So fix $\lambda \in \mathscr{P}$ with $|\lambda| \geq 1$ and assume that the thesis holds for all partitions such that $|\rho| < |\lambda|$. Since $(0) \notin S$, $|\rho| < |\lambda|$ and hence, by our induction hypothesis, $R_\rho(x)$ is a Λ-polynomial and $C(R_\rho) = (n|\rho| + k)/2$, for all partitions ρ appearing in the first sum on the RHS of (5). But, by hypothesis, $r_\mu(x)$ is a Λ-polynomial and $C(r_\mu) = n|\mu|/2$ for all $\mu \in S$. Therefore, by Theorem 1, $R_\rho(x) r_\mu(x)$ is a Λ-polynomial and

$$\begin{aligned} C(R_\rho r_\mu) &= \frac{n(|\rho| + |\mu|) + k}{2} \\ &= \frac{n|\lambda| + k}{2} \end{aligned}$$

for all ρ and μ appearing in the first sum on the RHS of (5). On the other hand, by our assumptions, all polynomials $u_\nu(x) f_{\nu\lambda}$ appearing in the second sum on the RHS of (5) are also Λ-polynomials and

$$\begin{aligned} C(u_\nu(x) f_{\nu\lambda}) &= \frac{n|\nu| + k}{2} \\ &= \frac{n|\lambda| + k}{2}. \end{aligned}$$

So all polynomials on the RHS of (5) are Λ-polynomials with center of symmetry equal to $(n|\lambda| + k)/2$. Therefore their sum, $R_\lambda(x)$, is also a Λ-polynomial with the same center of symmetry, as desired. This concludes the induction step and hence the proof. □

It may seem that Theorem 2 should be of limited use because it produces only *one* sequence of Λ-polynomials starting with *two* such sequences. However, as will be seen in the next section, in many applications the sequence $\{R_\lambda(q)\}_{\lambda\in\mathscr{P}}$ is much more complicated than the two starting sequences $\{u_\nu(q)\}_{\nu\in T}$ and $\{r_\mu(q)\}_{\mu\in S}$.

3. Applications

Our first application of Theorem 2 is the following.

Proposition 2. *For each partition λ define a polynomial $R_\lambda(q)$ by*

$$(6)\qquad \sum_\lambda R_\lambda(q) s_\lambda = \frac{1}{1 - \sum_{k\geq 2}(q + q^2 + \cdots + q^{k-1}) s_k}$$

Then $R_\lambda(q)$ is a Λ-polynomial and $C(R_\lambda) = |\lambda|/2$.

Proof. Take $a_\lambda = b_\lambda = c_\lambda = s_\lambda$ for $\lambda \in \mathscr{P}$ and let $S \stackrel{\text{def}}{=} \{(k): k \geq 2\}$, $r_\mu(x) \stackrel{\text{def}}{=} q+q^2+\cdots+q^{|\mu|-1}$ if $\mu \in S$, $T \stackrel{\text{def}}{=} \{(0)\}$, $\mu_{(0)}(x) \stackrel{\text{def}}{=} 1$. Then all the hypotheses of Theorem 2 are satisfied with $n = 1$ and $k = 0$, and the result follows. □

Proposition 2 verifies a conjecture of R. Stanley (see the remarks preceding Proposition 7.8 in [20]).

Our second application is closely related to the previous one.

Proposition 3. *For each partition* λ, *define a polynomial* $T_\lambda(q)$ *by*

$$\sum_\lambda T_\lambda(q)s_\lambda = \frac{\sum_{k\geq 1}(1+q+\cdots+q^{k-1})s_k}{1-\sum_{k\geq 2}(q+q^2+\cdots+q^{k-1})s_k}. \tag{7}$$

Then $T_\lambda(q)$ *is a* Λ*-polynomial and* $C(T_\lambda) = (|\lambda|-1)/2$.

Proof. Take a_λ, b_λ, c_λ, S and $r_\mu(q)$ as in the proof of Proposition 2 and let $T \stackrel{\text{def}}{=} \{(k): k \geq 1\}$, $u_\nu(q) \stackrel{\text{def}}{=} 1+q+q^2+\cdots+q^{|\nu|-1}$ if $\nu \in T$. Then all the hypotheses of Theorem 2 are satisfied with $n = 1$ and $k = -1$ and the result follows. □

Proposition 3 is equivalent to a result of R. Stanley (see Proposition 7.7 in [20]). However, our proof only uses standard results from the theory of symmetric functions while the sketch of proof given in [20] uses techniques from the theory of representations of the symmetric group and from algebraic geometry. To see that the two results are equivalent just observe that

$$\frac{\sum_{k\geq 1}(1+q+\cdots+q^{k-1})s_k}{1-\sum_{k\geq 2}(q+q^2+\cdots+q^{k-1})s_k} + 1 = \frac{\sum_{k\geq 0}s_k}{1-\sum_{k\geq 2}(q+q^2+\cdots+q^{k-1})s_k},$$

which is the definition used in Proposition 7.7 of [20].

4. Combinatorial consequences

In this section we look at some combinatorial properties of the polynomials studied in the last section. Following [7, §3], we define a map $R\colon \Lambda_{\mathbf{Q}[q]} \to \mathbf{Q}[q][[t]]$ by

$$R(p) \stackrel{\text{def}}{=} \sum_{n\geq 0}[x_1\cdots x_n](p)\frac{t^n}{n!} \tag{8}$$

where $p \in \Lambda$ and $[x_1\cdots x_n](p)$ denotes the coefficient of $x_1\cdots x_n$ in p (where $\prod_{i=1}^n x_i \stackrel{\text{def}}{=} 1$ if $n = 0$). It is then easy to see that R is a ring homomorphism. Also, from the combinatorial interpretation of the Schur functions (see, e.g., [9, Equation (5.12), p. 42] or [15]) it follows immediately that, for any partition λ,

$$R(s_\lambda) = f^\lambda \frac{t^{|\lambda|}}{|\lambda|!} \tag{9}$$

where f^λ denotes the number of standard tableaux of shape λ (see, e.g., [9, p. 5] for the definition of a standard tableau).

We may now prove the following result.

Proposition 4. *Let* $T_\lambda(q)$ *be the polynomials defined in Proposition* 3. *Then, for* $n \in \mathbf{P}$,

$$\sum_{\lambda \vdash n} f^\lambda T_\lambda(q) = \frac{1}{q} A_n(q), \tag{10}$$

where $A_n(q)$ *is the* n*th Eulerian polynomial.*

Proof. Multiplying both sides of (7) by q and then adding 1 gives

$$1 + \sum_{\lambda \in \mathscr{P}} qT_\lambda(q)s_\lambda = \frac{1 + qt + \sum_{k\geq 2} q^k s_k}{1 - \sum_{k\geq 2}(q + q^2 + \cdots + q^{k-1})s_k};$$

applying R on both sides and using (8) and (9) we obtain

$$\begin{aligned} 1 + \sum_{\lambda \in \mathscr{P}} qT_\lambda(q) f^\lambda \frac{t^{|\lambda|}}{|\lambda|!} &= \frac{e^{qt}}{1 - \frac{1}{1-q}(qe^t - q - e^{qt} + 1)} \\ &= \frac{(1-q)e^{qt}}{e^{qt} - qe^t} \\ &= \sum_{n\geq 0} A_n(q) \frac{t^n}{n!}, \end{aligned}$$

where, in the last equality, we have used a well known generating function for Eulerian polynomials (see, e.g., Equation (5i) on p. 244 of [5]). Equating the coefficients of t^n yields (10), as desired. □

Note that, by Proposition 3, the preceding result gives yet another proof of the well known fact that the Eulerian polynomials are symmetric and unimodal.

Now let $n \in \mathbf{P}$ and $\sigma \in S_n$ (where S_n is the symmetric group on n elements). An element $i \in [n]$ (where $[n] \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$) is called an *excedance* of σ if $\sigma(i) > i$; denote the number of excedances of σ by $e(\sigma)$. The permutation $\sigma \in S_n$ is called a *derangement* if $\sigma(i) \neq i$ for $i = 1, 2, \ldots, n$ (i.e., σ has no fixed points). We denote the set of all derangements of S_n by D_n, we define polynomials $d_n(q)$ by

$$d_n(q) \stackrel{\text{def}}{=} \sum_{\sigma \in D_n} q^{e(\sigma)}, \tag{11}$$

for $n \in \mathbf{P}$ (so that $d_1(q) = 0$), and let $d_0(q) \stackrel{\text{def}}{=} 1$. Since $d_n(1) = |D_n|$ we may consider $d_n(q)$ as a q-analogue of the derangement numbers. These are different, however, from other q-derangement numbers that have been previously considered in the literature (see, e.g., [6, 21]).

It is not hard to write down the exponential generating function for the polynomials $d_n(q)$.

Proposition 5. *Let $d_n(q)$ be the polynomials defined by* (11). *Then*

$$\sum_{n\geq 0} d_n(q)\frac{t^n}{n!} = \frac{1-q}{e^{tq}-qe^t}.$$

Proof. It is well known (see, e.g., [18, Proposition 1.3.12]) that $A_n(q) = \sum_{\sigma\in S_n} q^{e(\sigma)+1}$. Therefore, for $n \in \mathbf{P}$,

$$\begin{aligned}\frac{1}{q}A_n(q) &= \sum_{S\subseteq[n]}\sum_{\sigma\in D_{|s|}} q^{e(\sigma)}\\ &= \sum_{i=0}^{n}\binom{n}{i} d_i(q).\end{aligned}$$

It follows that

$$\begin{aligned}qe^t\sum_{n\geq 0} d_n(q)\frac{t^n}{n!} &= \sum_{n\geq 0} A_n(q)\frac{t^n}{n!} + q - 1\\ &= \frac{(1-q)e^{qt}}{e^{qt}-qe^t} + (q-1)\\ &= \frac{(1-q)qe^t}{e^{tq}-qe^t},\end{aligned}$$

where in the last equality we have again used Equation (5i) on p. 244 of [5], and the proof follows. □

We can now prove the following result.

Proposition 6. *Let $R_\lambda(q)$ be the polynomials defined in Proposition* 2. *Then, for $n \in \mathbf{N}$,*

$$\sum_{\lambda\vdash n} f^\lambda R_\lambda(q) = d_n(q).$$

Proof. Applying the homomorphism R defined by (8) to both sides of (6) yields

$$\begin{aligned}\sum_{\lambda\in\mathscr{P}} R_\lambda(q) f^\lambda\frac{t^{|\lambda|}}{|\lambda|!} &= \frac{1}{1-\sum_{k\geq 2}(q+q^2+\cdots+q^{k-1})\frac{t^k}{k!}}\\ &= \frac{1-q}{e^{tq}-qe^t}\\ &= \sum_{n\geq 0} d_n(q)\frac{t^n}{n!},\end{aligned}$$

and the proof follows. □

By Proposition 2, the following is an immediate consequence of the previous result.

Corollary 1. *For $n \in \mathbf{P}$, the polynomials $d_n(q)$ defined by (11) are symmetric and unimodal.*

It would be interesting to have a combinatorial proof of this result.

Even though the techniques presented in this paper cannot be used to prove it, we feel that the following stronger statement actually holds.

Conjecture. *For $n \in \mathbf{P}$, the polynomials $d_n(q)$ defined by (11) have only real zeros.*

The conjecture has been verified for $n \leq 14$. It is possible that the techniques used in [2] and [3] may be useful in attacking the above conjecture which, in fact, is closely related to one of the conjectures appearing in §3.4 of [2].

Given the results of Propositions 4 and 6 it is natural to ask for a combinatorial interpretation of the polynomials $T_\lambda(q)$ and $R_\lambda(q)$ themselves, a problem that had already been raised in [20]. Such a combinatorial interpretation has recently been found by J. Stembridge and will appear in a forthcoming paper of his.

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