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ON A CHARACTERIZATION OF TREES

WITOLD D. BULA AND E. D. TYMCHATYN

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ABSTRACT. We prove that a continuum X is a tree if and only if for each pair $K \subset L$ of its nondegenerate subcontinua some subcontinuum of K separates L.

The purpose of this note is to give a slight generalization and a somewhat simpler proof of a recent characterization of trees by Ward [1].

Theorem. *A continuum X is a tree if and only if for each pair of nondegenerate subcontinua K and L of X such that $K \subset L$ some subcontinuum of K separates L.*

A *continuum* is a compact connected Hausdorff space. A continuum X is a *tree* if every pair of distinct points of X can be separated by a third point of X. A continuum K in a space X is said to be a *continuum of convergence* of X if there is a net $\{K_\sigma : \sigma \in \Sigma\}$ of subcontinua of X converging to K and such that $K_\sigma \cap K = \emptyset$ and either $K_\sigma = K_\tau$ or $K_\sigma \cap K_\tau = \emptyset$ for all $\sigma, \tau \in \Sigma$. It is well known and easy to show that a continuum X that contains no continuum of convergence is locally connected.

The continuum X is said to be *unicoherent* if whenever $X = K \cup L$, where K and L are subcontinua of X, then $K \cap L$ is connected. We say that X is *hereditarily unicoherent* if each of its subcontinua is unicoherent.

Proof of the theorem. Assume that X is a tree. Let $K \subset L$ be nondegenerate subcontinua of X and let x and y be two points of K. If $z \in X$ separates x and y then $z \in K$. In particular, $\{z\}$ separates L.

Now assume that for each pair $K \subset L$ of nondegenerate subcontinua of X some subcontinuum of K separates L. We prove first that X is hereditarily unicoherent. Suppose that K and L are subcontinua of X such that $K \cap L = A \cup B$, where A and B are non-void disjoint closed sets. Let M be a subcontinuum of $K \cup L$ which is irreducible with respect to

$(*)$ both $M \cap K$ and $M \cap L$ are connected and $M \cap A \neq \emptyset \neq M \cap B$.

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By the Boundary Bumping Theorem (see [2, p.16]) there is a nondegenerate continuum $N \subset M \setminus L$. Let P be a subcontinuum of N and suppose that $M \setminus P = U \cup V$, where U and V are non-empty disjoint open sets in M. The set $M \cap L$ is connected, so we may suppose $M \cap L \subset U$. Hence, $(A \cup B) \cap M \subset U$ and $V \subset K \setminus L$. Also, $(M \cap K) \setminus P = (M \cap K \cap U) \cup (M \cap K \cap V)$, the union of two separated sets, so $(M \cap K \cap U) \cup P = (U \cup P) \cap K$ is connected. Hence, $U \cup P$ is a proper subcontinuum of M satisfying ($*$). This is a contradiction. Thus, N is a nondegenerate subcontinuum of M, no subcontinuum of which separates M contrary to the hypothesis of the theorem. Thus, every pair of subcontinua of X intersects in a connected set. It follows that for every pair (x, y) of distinct points of X, x and y are joined by a unique smallest continuum.

Suppose X contains a nondegenerate continuum of convergence K. Let $\{K_\sigma : \sigma \in \Sigma\}$ be a net of subcontinua in X converging to K and such that $K_\alpha \cap K = \varnothing$ and either $K_\alpha \cap K_\beta = \varnothing$ or $K_\alpha = K_\beta$ for $\alpha, \beta \in \Sigma$. For each $\sigma \in \Sigma$ let M_σ be a subcontinuum of X which irreducibly joins K and K_σ. By the above $M_\sigma \cap K$ is connected. By the irreducibility of M_σ no subcontinuum of $M_\sigma \cap K$ separates M_σ. By the hypothesis $M_\sigma \cap K = \{p_\sigma\}$ for some $p_\sigma \in K$. Let $\mathscr{U}$ be a finite cover of K by open sets in K whose closures do not contain K. For $U \in \mathscr{U}$ let $\Sigma(U) = \{\sigma \in \Sigma : p_\sigma \in U\}$. Since $\mathscr{U}$ is finite $\Sigma(W)$ is cofinal in Σ for some $W \in \mathscr{U}$. Without loss of generality we may assume $\Sigma(W) = \Sigma$. Let C be a nondegenerate continuum in $K \setminus clW$. Let $L = cl(\bigcup\{K_\sigma \cup M_\sigma : \sigma \in \Sigma\})$. It is easy to check that C contains no separating subcontinuum of L contrary to the hypothesis of the theorem. Hence, X is locally connected.

Finally we prove that X is a tree. Let x and y be two points of X. Let K be the smallest continuum in X from x to y. Let $z \in K \setminus \{x, y\}$. If z did not separate x from y in X it would follow from the local connectedness and the regularity of X and the Chaining Lemma (see [2, p.13]) that there is a continuum $L \subset X \setminus \{z\}$ such that $x, y \in L$. This would contradict the hereditary unicoherence of X. Hence, the point z separates x from y in X.

Corollary (Ward). *A continuum X is a tree if and only if for each pair of non-degenerate subcontinua K and L of X such that $K \subset L$ some point of K separates L.*

References

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Department of Mathematics, University of Saskatchewan, Saskatoon, Saskatchewan S7N 0W0 Canada