ON A CHARACTERIZATION OF TREES

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Abstract. We prove that a continuum \( X \) is a tree if and only if for each pair \( K \subset L \) of its nondegenerate subcontinua some subcontinuum of \( K \) separates \( L \).

The purpose of this note is to give a slight generalization and a somewhat simpler proof of a recent characterization of trees by Ward [1].

**Theorem.** A continuum \( X \) is a tree if and only if for each pair of nondegenerate subcontinua \( K \) and \( L \) of \( X \) such that \( K \subset L \) some subcontinuum of \( K \) separates \( L \).

A continuum is a compact connected Hausdorff space. A continuum \( X \) is a tree if every pair of distinct points of \( X \) can be separated by a third point of \( X \). A continuum \( K \) in a space \( X \) is said to be a continuum of convergence of \( X \) if there is a net \( \{K_\sigma : \sigma \in \Sigma\} \) of subcontinua of \( X \) converging to \( K \) and such that \( K_\sigma \cap K = \emptyset \) and either \( K_\sigma = K_\tau \) or \( K_\sigma \cap K_\tau = \emptyset \) for all \( \sigma, \tau \in \Sigma \). It is well known and easy to show that a continuum \( X \) that contains no continuum of convergence is locally connected.

The continuum \( X \) is said to be unicoherent if whenever \( X = K \cup L \), where \( K \) and \( L \) are subcontinua of \( X \), then \( K \cap L \) is connected. We say that \( X \) is hereditarily unicoherent if each of its subcontinua is unicoherent.

**Proof of the theorem.** Assume that \( X \) is a tree. Let \( K \subset L \) be nondegenerate subcontinua of \( X \) and let \( x \) and \( y \) be two points of \( K \). If \( z \in X \) separates \( x \) and \( y \) then \( z \in K \). In particular, \( \{z\} \) separates \( L \).

Now assume that for each pair \( K \subset L \) of nondegenerate subcontinua of \( X \) some subcontinuum of \( K \) separates \( L \). We prove first that \( X \) is hereditarily unicoherent. Suppose that \( K \) and \( L \) are subcontinua of \( X \) such that \( K \cap L = A \cup B \), where \( A \) and \( B \) are non-void disjoint closed sets. Let \( M \) be a subcontinuum of \( K \cup L \) which is irreducible with respect to

\[ (*) \quad \text{both } M \cap K \text{ and } M \cap L \text{ are connected and } M \cap A \neq \emptyset \neq M \cap B. \]

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By the Boundary Bumping Theorem (see [2, p.16]) there is a nondegenerate continuum \( N \subset M \setminus L \). Let \( P \) be a subcontinuum of \( N \) and suppose that \( M \setminus P = U \cup V \), where \( U \) and \( V \) are non-empty disjoint open sets in \( M \). The set \( M \cap L \) is connected, so we may suppose \( M \cap L \subset U \). Hence, \((A \cup B) \cap M \subset U \) and \( V \subset K \setminus L \). Also, \((M \cap K) \setminus P = (M \cap K \cap U) \cup (M \cap K \cap V) \), the union of two separated sets, so \((M \cap K \cap U) \cup P = (U \cup P) \cap K \) is connected. Hence, \( U \cup P \) is a proper subcontinuum of \( M \) satisfying (\(*\)). This is a contradiction. Thus, \( N \) is a nondegenerate subcontinuum of \( M \), no subcontinuum of which separates \( M \) contrary to the hypothesis of the theorem. Thus, every pair of subcontinua of \( X \) intersects in a connected set. It follows that for every pair \( (x,y) \) of distinct points of \( X \), \( x \) and \( y \) are joined by a unique smallest continuum.

Suppose \( X \) contains a nondegenerate continuum of convergence \( K \). Let \( \{K_\sigma : \sigma \in \Sigma \} \) be a net of subcontinua in \( X \) converging to \( K \) and such that \( K_\alpha \cap K = \emptyset \) and either \( K_\alpha \cap K_\beta = \emptyset \) or \( K_\alpha = K_\beta \) for \( \alpha, \beta \in \Sigma \). For each \( \sigma \in \Sigma \) let \( M_\sigma \) be a subcontinuum of \( X \) which irreducibly joins \( K \) and \( K_\sigma \). By the above \( M_\sigma \cap K \) is connected. By the irreducibility of \( M_\sigma \) no subcontinuum of \( M_\sigma \cap K \) separates \( M_\sigma \). By the hypothesis \( M_\sigma \cap K = \{p_\sigma\} \) for some \( p_\sigma \in K \). Let \( \mathcal{U} \) be a finite cover of \( K \) by open sets in \( K \) whose closures do not contain \( K \). For \( U \in \mathcal{U} \) let \( \Sigma(U) = \{\sigma \in \Sigma : p_\sigma \in U \} \). Since \( \mathcal{U} \) is finite \( \Sigma(W) \) is cofinal in \( \Sigma \) for some \( W \in \mathcal{U} \). Without loss of generality we may assume \( \Sigma(W) = \Sigma \). Let \( C \) be a nondegenerate continuum in \( K \setminus \text{cl}(W) \). Let \( L = \text{cl}(\bigcup\{K_\sigma \cup M_\sigma : \sigma \in \Sigma\}) \). It is easy to check that \( C \) contains no separating subcontinuum of \( L \) contrary to the hypothesis of the theorem. Hence, \( X \) is locally connected.

Finally we prove that \( X \) is a tree. Let \( x \) and \( y \) be two points of \( X \). Let \( K \) be the smallest continuum in \( X \) from \( x \) to \( y \). Let \( z \in K \setminus \{x, y\} \). If \( z \) did not separate \( x \) from \( y \) in \( X \) it would follow from the local connectedness and the regularity of \( X \) and the Chaining Lemma (see [2, p.13]) that there is a continuum \( L \subset X \setminus \{z\} \) such that \( x, y \in L \). This would contradict the hereditary unicoherence of \( X \). Hence, the point \( z \) separates \( x \) from \( y \) in \( X \).

**Corollary** (Ward). A continuum \( X \) is a tree if and only if for each pair of nondegenerate subcontinua \( K \) and \( L \) of \( X \) such that \( K \subset L \) some point of \( K \) separates \( L \).

**References**


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