ON A CHARACTERIZATION OF TREES

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Abstract. We prove that a continuum $X$ is a tree if and only if for each pair $K \subset L$ of its nondegenerate subcontinua some subcontinuum of $K$ separates $L$.

The purpose of this note is to give a slight generalization and a somewhat simpler proof of a recent characterization of trees by Ward [1].

Theorem. A continuum $X$ is a tree if and only if for each pair of nondegenerate subcontinua $K$ and $L$ of $X$ such that $K \subset L$ some subcontinuum of $K$ separates $L$.

A continuum is a compact connected Hausdorff space. A continuum $X$ is a tree if every pair of distinct points of $X$ can be separated by a third point of $X$. A continuum $K$ in a space $X$ is said to be a continuum of convergence of $X$ if there is a net $\{K_\sigma : \sigma \in \Sigma\}$ of subcontinua of $X$ converging to $K$ and such that $K_\sigma \cap K = \emptyset$ and either $K_\sigma = K_\tau$ or $K_\sigma \cap K_\tau = \emptyset$ for all $\sigma, \tau \in \Sigma$. It is well known and easy to show that a continuum $X$ that contains no continuum of convergence is locally connected.

The continuum $X$ is said to be unicoherent if whenever $X = K \cup L$, where $K$ and $L$ are subcontinua of $X$, then $K \cap L$ is connected. We say that $X$ is hereditarily unicoherent if each of its subcontinua is unicoherent.

Proof of the theorem. Assume that $X$ is a tree. Let $K \subset L$ be nondegenerate subcontinua of $X$ and let $x$ and $y$ be two points of $K$. If $z \in X$ separates $x$ and $y$ then $z \in K$. In particular, $\{z\}$ separates $L$.

Now assume that for each pair $K \subset L$ of nondegenerate subcontinua of $X$ some subcontinuum of $K$ separates $L$. We prove first that $X$ is hereditarily unicoherent. Suppose that $K$ and $L$ are subcontinua of $X$ such that $K \cap L = A \cup B$, where $A$ and $B$ are non-void disjoint closed sets. Let $M$ be a subcontinuum of $K \cup L$ which is irreducible with respect to

$$ (*) \quad \text{both } M \cap K \text{ and } M \cap L \text{ are connected and } M \cap A \neq \emptyset \neq M \cap B. $$

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By the Boundary Bumping Theorem (see [2, p.16]) there is a nondegenerate continuum $N \subset M \setminus L$. Let $P$ be a subcontinuum of $N$ and suppose that $M \setminus P = U \cup V$, where $U$ and $V$ are non-empty disjoint open sets in $M$. The set $M \cap L$ is connected, so we may suppose $M \cap L \subset U$. Hence, $(A \cup B) \cap M \subset U$ and $V \subset K \setminus L$. Also, $(M \cap K) \setminus P = (M \cap K \cap U) \cup (M \cap K \cap V)$, the union of two separated sets, so $(M \cap K \cap U) \cup P = (U \cup P) \cap K$ is connected. Hence, $U \cup P$ is a proper subcontinuum of $M$ satisfying $(\ast)$. This is a contradiction. Thus, $N$ is a nondegenerate subcontinuum of $M$, no subcontinuum of which separates $M$ contrary to the hypothesis of the theorem. Thus, every pair of subcontinua of $X$ intersects in a connected set. It follows that for every pair $(x, y)$ of distinct points of $X$, $x$ and $y$ are joined by a unique smallest continuum.

Suppose $X$ contains a nondegenerate continuum of convergence $K$. Let\{K$_{\alpha}$ : $\alpha \in \Sigma$\} be a net of subcontinua in $X$ converging to $K$ and such that $K_{\alpha} \cap K = \emptyset$ and either $K_{\alpha} \cap K_{\beta} = \emptyset$ or $K_{\alpha} = K_{\beta}$ for $\alpha, \beta \in \Sigma$. For each $\sigma \in \Sigma$ let $M_{\sigma}$ be a subcontinuum of $X$ which irreducibly joins $K$ and $K_{\sigma}$. By the above $M_{\sigma} \cap K$ is connected. By the irreducibility of $M_{\sigma}$ no subcontinuum of $M_{\sigma} \cap K$ separates $M_{\sigma}$. By the hypothesis $M_{\sigma} \cap K = \{p_{\sigma}\}$ for some $p_{\sigma} \in K$. Let $\mathcal{U}$ be a finite cover of $K$ by open sets in $K$ whose closures do not contain $K$. For $U \in \mathcal{U}$ let $\Sigma(U) = \{\sigma \in \Sigma : p_{\sigma} \in U\}$. Since $\mathcal{U}$ is finite $\Sigma(W)$ is cofinal in $\Sigma$ for some $W \in \mathcal{U}$. Without loss of generality we may assume $\Sigma(W) = \Sigma$. Let $C$ be a nondegenerate continuum in $K \setminus c(W)$. Let $L = c(\bigcup\{K_{\sigma} \cup M_{\sigma} : \sigma \in \Sigma\})$. It is easy to check that $C$ contains no separating subcontinuum of $L$ contrary to the hypothesis of the theorem. Hence, $X$ is locally connected.

Finally we prove that $X$ is a tree. Let $x$ and $y$ be two points of $X$. Let $K$ be the smallest continuum in $X$ from $x$ to $y$. Let $z \in K \setminus \{x, y\}$. If $z$ did not separate $x$ from $y$ in $X$ it would follow from the local connectedness and the regularity of $X$ and the Chaining Lemma (see [2, p.13]) that there is a continuum $L \subset X \setminus \{z\}$ such that $x, y \in L$. This would contradict the hereditary unicoherence of $X$. Hence, the point $z$ separates $x$ from $y$ in $X$.

**Corollary** (Ward). A continuum $X$ is a tree if and only if for each pair of nondegenerate subcontinua $K$ and $L$ of $X$ such that $K \subset L$ some point of $K$ separates $L$.

**References**


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