PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 4, April 1990

NEUMANN EIGENVALUE ESTIMATE ON A COMPACT RIEMANNIAN MANIFOLD

ROGER CHEN

(Communicated by Jonathan M. Rosenberg)

ABSTRACT. In their article, P. Li and S. T. Yau give a lower bound of the first Neumann eigenvalue in terms of geometrical invariants for a compact Riemannian manifold with convex boundary. The purpose of this paper is to generalize their result to a compact Riemannian manifold with possibly nonconvex boundary.

1. INTRODUCTION

Let M^n be an n-dimensional compact Riemannian manifold with boundary ∂M. In local coordinates $(x^1, x^2, \dots, x^n)$, the Riemannian metric is given by

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j. \tag{1.1}$$

One defines on M a second order elliptic differential operator by

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right), \tag{1.2}$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$, which is known as the Laplace operator. The purpose of this paper is the study of eigenvalues of the Laplace operator. More specifically, we study the following problem:

Assume that $\partial M \neq \emptyset$, we adopt an "interior rolling ε-ball" condition on ∂M to consider the following Neumann eigenvalue problem on M^n:

$$\begin{cases} \Delta h &= -\eta h, \\ \frac{\partial h}{\partial \nu} &\equiv 0 \text{ on } \partial M, \end{cases} \tag{1.3}$$

where ν is the unit outward normal vector to the boundary ∂M.

Definition 1.1. Let ∂M be the boundary of a compact Riemannian manifold M^n. Then ∂M satisfies the "interior rolling ε-ball" condition if for each point

Received by the editors December 8, 1988 and, in revised form, April 17, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53-XX, 58G20, 58G25.

$p \in \partial M$ there is a geodesic ball $B_q(\varepsilon/2)$, centered at $q \in M$ with radius $\varepsilon/2$, such that

$$p = B_q(\varepsilon/2) \cap \partial M \quad \text{and} \quad B_q(\varepsilon/2) \subset M.$$

It is well known that the set of eigenvalues $\{\eta_k\}$ of (1.3) are nonnegative and can be arranged in a nondecreasing order of magnitude as follows:

$$0 = \eta_0 < \eta_1 \le \eta_3 \le \cdots \le \eta_m \le \cdots .$$

By the compactness of M^n, it is known that those functions which satisfy (1.3) with eigenvalue η_0 are constants. The first nonzero eigenvalue η_1 in the problem (1.3) is hence characterized as the optimal constant in the Poincaré inequality:

$$\eta_1 \int f^2 \le \int |\nabla f|^2 \tag{1.4}$$

for all $f \in H_1^2(M)$ such that $\int_M f = 0$. Due to the importance of Poincaré inequality for analysis on manifolds, one wishes to obtain optimal quantitative estimates for the first eigenvalue η_1 from below in terms of geometric elements. Classically, for domains in $\mathbf{R}^n$, lower estimates for η_1 were established by Payne-Weinberger [4] and Payne-Stakgold [3], etc. For general compact manifolds with convex boundary, the lower estimates of η_1 were obtained by Li-Yau [2]. Using a method similar to that of Li-Yau [2], we have the following:

Theorem 1.1. *Let M^n be a compact Riemannian manifold with boundary ∂M. Let ∂M satisfy the "interior rolling ε-ball" condition. Let K and H be nonnegative constants such that the Ricci curvature Ric_M of M is bounded below by $-K$ and the second fundamental form elements of ∂M is bounded below by $-H$. By choosing ε "small", we have*

$$\frac{1}{(1+H)^2}\left[\frac{1-\alpha^2}{4(n-1)d^2}B^2 - C\right]\exp(-B) \le \eta_1 \tag{1.5}$$

where α and ε are positive constants less than 1.

$$d = \text{ diameter of } M^n,$$

$$B = 1 + [1 + \frac{4(n-1)d^2C)}{1-\alpha^2}]^{1/2},$$

$$C = (1+H)C_1 + \frac{[(2n-3)^2 + (4n-5)\alpha^2]H^2}{(n-1)\varepsilon^2\alpha^2} + (H+1)^2K,$$

and

$$C_1 = \frac{2(n-1)H(3H+1)(H+1)}{\varepsilon} + \frac{H+H^2}{\varepsilon^2}.$$

Remark 1. When the boundary ∂M is convex, our estimate implies the estimate, obtained by Li-Yau [2, Theorem 9].

Remark 2. In our estimate, the choice of ε depends on the upper bound of the sectional curvature near the boundary. We do not know whether we can determine the upper bound of ε without using the curvature bound near the boundary. The upper bound of ε is given by (2.16) and (2.17).

In §2, we shall give a gradient estimate which is essential in a proof of the main result. In §4, we shall give a proof of Theorem 1.1. In §3, a counterexample is given which will demonstrate that the "interior rolling ε-ball" condition is definitely necessary for η_1 being bounded away from 0.

2. A GRADIENT ESTIMATE

Throughout this section, M^n is assumed to be an n-dimensional compact Riemannian manifold with boundary ∂M. Let K, H be nonnegative constants which were defined in §1. Let f be a function which satisfies (1.3) with $\eta = \eta_1$; i.e., f is an eigenfunction of (1.3) with eigenvalue η_1.

In this section, our goal is the study of the solution of equation (1.3) using maximal principle.

Let us first recall some general facts concerning a Riemannian manifold. Let $\{e_i\}$ be a local frame field of a Riemannian manifold M^n and $\{\omega_i\}$ be the corresponding dual frame field. Then the structure equations of M^n are given by

$$d\omega_j = \sum_{i=1}^{n} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} = -\omega_{ji}, \tag{2.1}$$

$$d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \tfrac{1}{2} \sum_{k,l}^{n} R_{ijkl}\omega_l \wedge \omega_k. \tag{2.2}$$

For any C^2-function $p(x)$ defined on M^n, we may define its gradient and Hessian by the following formulas:

$$dp = \sum_{i=1}^{n} p_i \omega_i, \tag{2.3}$$

$$\sum_{j=1}^{n} p_{ij}\omega_j = dp_i + \sum_{j=1}^{n} p_j \omega_{ji}; \tag{2.4}$$

and the covariant derivatives of p_{ij} are defined by

$$\sum_{k=1}^{n} p_{ijk}\omega_k = dp_{ij} + \sum_{k=1}^{n} p_{kj}\omega_{ki} + \sum_{k=1}^{n} p_{ik}\omega_{kj}. \tag{2.5}$$

By exteriorly differentiating (2.4), we get the following commutational formula:

$$p_{ijk} - p_{ikj} = \sum_{l=1}^{n} p_l R_{lijk}. \tag{2.6}$$

We have the following:

Theorem 2.1. *Let M^n be an n-dimensional compact Riemannian manifold with boundary ∂M satisfying the "interior rolling ε-ball" condition. Let f be a solution of equation* (1.4) *with $\eta = \eta_1$. If $\mu > 1$ is any constant, and ε is "small", then*

(2.7)
$$\frac{|\nabla f|^2}{(\mu \sup f - f)^2} \le \max\left\{\frac{4(n-1)}{1-\alpha^2}\left[C + (H+1)^2\eta_1 \left\|\frac{\mu \sup f}{\mu \sup f - f}\right\|_\infty\right], \frac{\sqrt{8}(H+1)^2}{\sqrt{1-\alpha^2}}\eta_1\left\|\frac{f}{\mu\sup f - f}\right\|_\infty\right\},$$

where α and ε are positive constants less than 1, *and*

$$\begin{aligned} C &= (1+H)C_1 + \frac{[(2n-3)^2 + (4n-5)\alpha^2]H^2}{(n-1)\varepsilon^2\alpha^2} + (1+H)^2 K, \\ C_1 &= \frac{2(n-1)H(3H+1)(1+H)}{\varepsilon} + \frac{H+H^2}{\varepsilon^2}. \end{aligned} \tag{2.8}$$

Remark 1. When the boundary ∂M is convex, our theorem implies the gradient estimate obtained in Li-Yau [2, Theorem 3].

Remark 2. In our estimate, ε is chosen to be a positive constant less than 1 and is dependent on the upper bound of the sectional curvature of the manifold near the boundary. The upper bound of ε is given by (2.16) and (2.17).

Proof. Let $\psi(r)$ be a nonnegative C^2 function defined on $[0,\infty)$ such that

$$\psi(r)\begin{cases} \le H & \text{if } r \in [0,\tfrac{1}{2}), \\ = H & \text{if } r \in [1,\infty), \end{cases}$$

with

$$\psi(0) = 0, \qquad 2H \ge \psi'(r) \ge 0, \qquad \psi'(0) = H$$

and

$$\psi''(r) \ge -H.$$

Let

$$\phi(x) = \psi\left(\frac{r(x)}{\varepsilon}\right),$$

where $r(x)$ denotes distance between boundary ∂M and $x \in M$. For $\mu > 1$, we define the function

$$G(x) = (1+\phi)^2 \frac{|\nabla f|^2}{(\mu \sup f - f)^2}. \tag{2.9}$$

By the compactness of M, there is a point $p \in M$ such that G achieves its supremum. We may assume that $G(p) > 0$, or else the theorem follows trivially. Suppose that p is a boundary point of ∂M. At p be may choose an orthonormal frame $e_1, \cdots, \varepsilon_n$ such that $\varepsilon_n = \frac{\partial}{\partial \nu}$, where ν is the unit outward normal vector to ∂ M. Then we have

$$0 \le \frac{\partial G}{\partial \nu}(p).$$

This gives

$$\begin{aligned} 0 &\le \frac{\partial \phi / \partial \nu}{1+\phi} + \frac{\sum_{i=1}^{n} f_i f_{i\nu}}{|\nabla f|^2} - \frac{f\nu}{\mu \sup f - f} \\ &= -\frac{H}{\varepsilon} + \frac{\sum_{i=1}^{n-1} f_i f_{i\nu}}{|\nabla f|^2}. \end{aligned} \tag{2.10}$$

If h_{ij} are the second fundamental form elements of ∂M, then by a direct computation one shows that

$$f_{i\nu} = -\sum_{j=1}^{n-1} h_{ij} f_j \quad \text{for } 1 \le i \le n-1, \tag{2.11}$$

where we used the fact that $f_\nu \equiv 0$ on ∂M. Together with (2.9), we have

$$\begin{aligned} 0 &\le -\frac{H}{\varepsilon} - \frac{\sum_{i,j=1}^{n-1} h_{ij} f_i f_j}{|\nabla f|^2} \\ &\le -\frac{H}{\varepsilon} + H \\ &< 0, \end{aligned}$$

which is a contradiction, as we choose ε to be smaller than 1. Hence $G(x)$ cannot attain its maximum at the boundary point. Therefore p has to be an interior point of M. Hence at p

$$\nabla G = 0 \tag{2.12}$$

and

$$\Delta G \le 0. \tag{2.13}$$

This gives

$$0 = \frac{\psi' r_j}{\varepsilon(1+\phi)} + \frac{\sum_{i=1}^{n} f_i f_{ij}}{|\nabla f|^2} + \frac{f_j}{\mu \sup f - f}, \tag{2.14}$$

and

$$\begin{aligned} 0 &\ge \frac{\Delta \phi}{1+\phi} - \frac{(\psi')^2}{\varepsilon^2 (1+\phi)^2} + \left(\sum_{i,j=1}^{n} f_{ij}^2 + \sum_{i,j=1}^{n} f_i f_{ijj} \right) \Big/ |\nabla f|^2 \\ &\quad - \frac{2 \sum_{j=1}^{n} (\sum_{i=1}^{n} f_i f_{ij})^2}{|\nabla f|^4} + \frac{\Delta f}{\mu \sup f - f} + \frac{|\nabla f|^2}{(\mu \sup f - f)^2}. \end{aligned} \tag{2.15}$$

To compute $\Delta\phi$, we let $\partial M(\varepsilon) = \{x \in M | r(x) \leq \varepsilon\}$ and K_ε be the upper bound of the sectional curvature in $\partial M(\varepsilon)$. We may choose ε to be small so that

$$\sqrt{K_\varepsilon} \tan\left(\varepsilon\sqrt{K_\varepsilon}\right) \leq \frac{H}{2} + \frac{1}{2} \tag{2.16}$$

and

$$\frac{H}{\sqrt{K_\varepsilon}} \tan\left(\varepsilon\sqrt{K_\varepsilon}\right) \leq \frac{1}{2}. \tag{2.17}$$

By using an index comparison theorem in Riemannian geometry [5, p. 347], one can show that if $x \in \partial M(\varepsilon)$, we have

$$\begin{aligned} \Delta r &\geq -(n-1)\frac{\varepsilon H + \varepsilon\sqrt{K_\varepsilon}\tan(\varepsilon\sqrt{K_\varepsilon})}{\varepsilon - \varepsilon\frac{H}{\sqrt{K_\varepsilon}}\tan(\varepsilon\sqrt{K_\varepsilon})} \\ &\geq -(n-1)(3H+1). \end{aligned}$$

Then we have

$$\begin{aligned} \Delta\phi &= \frac{1}{\varepsilon}\psi'\Delta r + \frac{1}{\varepsilon^2}\psi''|\nabla r|^2 \\ &\geq -\frac{2(n-1)H(3H+1)}{\varepsilon} - \frac{H}{\varepsilon^2} \\ &= -C_1. \end{aligned} \tag{2.18}$$

At p, we may choose an orthonormal frame $\{e_i\}$ such that $f_1(p) = |\nabla f|(p)$. By using (2.6) and (2.14), we also have, at p,

$$f_{j1j} - f_{jj1} = \sum_{l=1}^{n} f_l R_{lj1j}$$

and

$$f_{1j} = -\frac{\psi' f_1 r_j}{\varepsilon(1+\phi)} - \frac{f_1 f_j}{\mu \sup f - f}.$$

Substituting these and (2.18) into (2.15), we have

$$\begin{aligned} 0 &\geq -\frac{C_1}{1+\phi} - \frac{[1+r_1^2](\psi')^2}{\varepsilon^2(1+\phi)^2} - \frac{2\psi' f_1 r_1}{\varepsilon(1+\phi)(\mu\sup f - f)} \\ &\quad + \frac{\sum_{i>1} f_{ii}^2}{f_1^2} + Ric_{11} - \frac{\eta_1 \mu \sup f}{\mu \sup f - f}, \\ &\geq -\frac{C_1}{1+\phi} - \frac{2(\psi')^2}{\varepsilon^2(1+\phi)^2} - \frac{2\psi' f_1 r_1}{\varepsilon(1+\phi)(\mu\sup f - f)} \\ &\quad + \frac{\sum_{i>1} f_{ii}^2}{f_1^2} + Ric_{11} - \frac{\eta_1 \mu \sup f}{\mu \sup f - f}. \end{aligned} \tag{2.19}$$

It is also clear that

$$\sum_{i>1} f_{ii}^2 \geq \frac{1}{n-1}\left(\sum_{i>1} f_{ii}\right)^2 \geq \frac{f_{11}^2}{2(n-1)} - \frac{(\Delta f)^2}{n-1}. \tag{2.20}$$

Then we have

$$\sum_{i>1} f_{ii}^2 \geq \frac{f_1^4}{2(n-1)(\mu \sup f - f)^2} + \frac{f_1^3 \psi' r_1}{\varepsilon(n-1)(1+\phi)(\mu \sup f - f)} + \frac{(\psi')^2 f_1^2 r_1^2}{2\varepsilon^2(n-1)(1+\phi)^2} - \frac{\eta_1^2 f^2}{n-1}. \tag{2.21}$$

Substituting (2.21) into (2.19), we have

$$0 \geq \frac{f_1^2}{2(n-1)(\mu \sup f - f)^2} - \frac{(2n-3) f_1 \psi' r_1}{\varepsilon(n-1)(1+\phi)(\mu \sup f - f)} - \frac{C_1}{1+\phi} - \frac{4(n-1)(\psi')^2 - (\psi')^2 r_1^2}{2\varepsilon^2(n-1)(1+\phi)^2} - K - \frac{\eta_1 \mu \sup f}{\mu \sup f - f} - \frac{\eta_1^2 f^2}{(n-1) f_1^2}. \tag{2.22}$$

It is clear that

$$\frac{\alpha^2 f_1^2}{2(n-1)(\mu \sup f - f)^2} - \frac{(2n-3) f_1 \psi' r_1}{\varepsilon(n-1)(1+\phi)(\mu \sup f - f)}$$

$$\geq -\frac{(2n-3)^2(\psi')^2(r_1)^2}{2\varepsilon^2\alpha^2(n-1)(1+\phi)^2}, \geq -\frac{(2n-3)^2(\psi')^2}{2\varepsilon^2\alpha^2(n-1)(1+\phi)^2}.$$

Substituting this into (2.22), we have

$$\begin{aligned} 0 &\geq \left(\frac{1-\alpha^2}{2(n-1)}\right)\frac{f_1^2}{(\mu \sup f - f)^2} + \frac{[\alpha^2 - (2n-3)^2](\psi')^2(r_1)^2}{2\varepsilon^2\alpha^2(n-1)(1+\phi)^2} \\ &\quad - \frac{2(\psi')^2}{\varepsilon^2(1+\phi)^2} - \frac{C_1}{1+\phi} - K - \frac{\eta_1 \mu \sup f}{\mu \sup f - f} - \frac{\eta_1^2 f^2}{(n-1) f_1^2}, \\ &\geq \left(\frac{1-\alpha^2}{2(n-1)}\right)\frac{f_1^2}{(\mu \sup f - f)^2} - \frac{[(2n-3)^2 - \alpha^2](\psi')^2}{2\varepsilon^2\alpha^2(n-1)(1+\phi)^2} \\ &\quad - \frac{2(\psi')^2}{\varepsilon^2(1+\phi)^2} - \frac{C_1}{1+\phi} - K - \frac{\eta_1 \mu \sup f}{\mu \sup f - f} - \frac{\eta_1^2 f^2}{(n-1) f_1^2}. \end{aligned} \tag{2.23}$$

Hence

(2.24)

$$0 \geq \frac{(1-\alpha^2)f_1^2}{2(n-1)(\mu \sup f - f)^2} - \left[\frac{[(2n-3)^2+(4n-5)\alpha^2](\psi')^2}{2\varepsilon^2\alpha^2(n-1)(1+\phi)^2} + \frac{C_1}{1+\phi} + K + \frac{\eta_1 \mu \sup f}{\mu \sup f - f}\right] - \frac{\eta_1^2 f^2}{(n-1)f_1^2}.$$

Multiplying through by $(1+\phi)^4 \frac{f_1^2}{(\mu \sup f - f)^2}$, (2.24) becomes

(2.25)

$$\begin{aligned} 0 &\geq \frac{1-\alpha^2}{2(n-1)}G^2 - \left[\frac{[(2n-3)^2+(4n-5)\alpha^2](\psi')^2}{2\varepsilon^2\alpha^2(n-1)} + (1+\phi)C_1 + (1+\phi)^2 K + \frac{\eta_1\mu \sup f(1+\phi)^2}{\mu \sup f - f}\right] G - \frac{\eta_1^2 f^2(1+\phi)^4}{(n-1)(\mu \sup f - f)^2} \\ &\geq \frac{1-\alpha^2}{2(n-1)}G^2 - \left[\frac{2[(2n-3)^2+(4n-5)\alpha^2]H^2}{\varepsilon^2\alpha^2(n-1)} + (1+H)C_1 + (1+H)^2 K + \frac{(1+H)^2\eta_1\mu \sup f}{\mu \sup f - f}\right] G - \frac{(1+H)^4\eta_1^2 f^2}{(n-1)(\mu \sup f - f)^2}. \end{aligned}$$

This implies Theorem 2.1.

3. A counterexample

In this section, we shall show that the "interior rolling ε-ball" condition is necessary for η_1 being bounded away from zero. We consider the following well-known example of E. Calabi [1]. For the sake of completeness, we will describe the example here.

Example 3.1. Let $\Omega \subset \mathbf{R}^2$ be a domain as in Figure 1. The rectangle connecting the two disks is to be thought of as having fixed length l and variable width w.

Let f be a function which is equal to c on the right-hand disk, $-c$ on the left-hand disk and changes linearly from c to $-c$ across the rectangle. (c is chosen to that $\int_\Omega f^2 = 1$.) Then $\int_\Omega f = 0$ and

$$\|\nabla f\| = \begin{cases} 0 & \text{on the disks}, \\ \frac{2c}{l} & \text{on the rectangle.} \end{cases}$$

It is clear that

$$\begin{aligned} \eta_1 &= \inf_{h \in H_1^2(\Omega) \text{ such that } \int_\Omega h = 0} \frac{\int_\Omega |\nabla h|^2}{\int_\Omega h^2} \\ &\leq \int_\Omega |\nabla f|^2 \\ &= \frac{2c}{l} wl \to 0, \quad \text{as } w \to 0. \end{aligned}$$

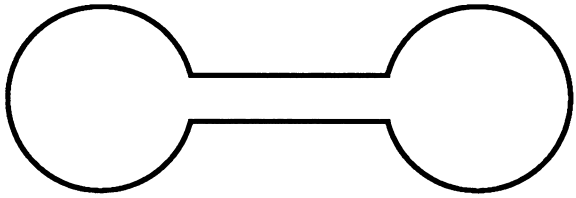

FIGURE 1. DUMBBELL

This example shows that in bounding η_1 from below, it is necessary to consider the "interior rolling ε-ball" condition.

4. PROOF OF THEOREM 1.1

In this section, we let f be a function which satisfies (1.3) with $\eta = \eta_1$; i.e., f is an eigenfunction of (1.3) with eigenvalue η_1. We denote N the nodal set of f; i.e., $N = \{x \in \overline{M} | f(x) = 0\}$.

Proof. From Theorem 2.1, we know that

$$\frac{|\nabla f|}{\mu \sup f - f} \leq \max\left\{ \frac{\sqrt{4(n-1)}}{\sqrt{1-\alpha^2}} \left[C + \frac{(H+1)^2 \eta_1 \mu}{\mu - 1}\right]^{1/2}, \frac{\sqrt{S}(H+1)}{\sqrt{1-\alpha^2}} \frac{\eta_1^{1/2}}{(\mu-1)^{1/2}} \right\} \tag{4.1}$$

for any constant $\mu > 1$.

However, since f satisfies

$$\int_M f = 0 \tag{4.2}$$

and

$$f \not\equiv 0, \tag{4.3}$$

this implies that the nodal set N of f divides $\overline{M}$ into two parts. If $x \in \overline{M}$ is a point where f achieves its supremum and γ is a shortest geodesic joining x and N, then γ has length at most diameter of M. Integrating (4.1) along γ, we have

$$\begin{aligned} \log \frac{\mu}{\mu - 1} &\leq \int_\gamma \frac{|\nabla f|}{\mu \sup f - f} \\ &\leq \frac{4\sqrt{(n-1)}}{1-\alpha^2} \left[C + \frac{(H+1)^2 \mu \eta_1}{\mu - 1}\right]^{\frac{1}{2}} d. \end{aligned} \tag{4.4}$$

Hence

$$\frac{\mu - 1}{(H+1)^2\mu}\left[\frac{1-\alpha^2}{4(n-1)d^2}\left(\log\frac{\mu}{\mu-1}\right)^2 - C\right] \le \eta_1. \tag{4.5}$$

It is clear that the left-hand side can be made to be positive by choosing μ close enough to 1. The theorem is then proved by maximizing (4.5) with

$$\frac{\mu}{\mu - 1} = \exp\left[1 + \left(1 + \frac{4(n-1)d^2}{1-\alpha^2}C\right)^{\frac{1}{2}}\right]. \tag{4.6}$$

This proves Theorem 1.1.

Acknowledgments

The author would like to express his gratitude to Professor Peter Li and Professor Andrejs Treibergs for their many helpful suggestions and for their encouragement. He is especially indebted to Professor Peter Li, under whose advice this paper was completed.

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Department of Mathematics, University of Toledo, Toledo, Ohio 43606

Current address: Department of Mathematics, University of Southern California, Los Angeles, California 90089