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ON DUAL BANACH ALGEBRAS

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ABSTRACT. Let A be a semisimple Banach algebra with $\|\cdot\|$, which is a dense subalgebra of a semisimple Banach algebra B with $|\cdot|$ such that $\|\cdot\|$ majorizes $|\cdot|$ on A. The purpose of this paper is to investigate the dual property between the algebras A and B. Some well-known results follow from this paper.

1. INTRODUCTION

Let A be a semisimple Banach algebra with norm $\|\cdot\|$ which is a dense subalgebra of a semisimple Banach algebra B with $|\cdot|$ such that $\|\cdot\|$ majorizes $|\cdot|$ on A. The purpose of this paper is to investigate the dual property between the algebras A and B.

It is shown that if A is a dual algebra, then B is a dual algebra if and only if, $R = cl_B(R \cap A)$, for any proper closed right (left) ideal R of B. On the other hand, if B is a dual algebra, then A is a dual algebra if and only, for any proper closed right (left) ideal N of A, $N = cl_B(N) \cap A$ and for any proper closed right (left) ideal R of B, $R = cl_B(R \cap A)$. If A is a two-sided ideal of B and B has a bounded right approximate identity and a bounded left approximate identity, then we show that A is a dual algebra if and only if B is a dual algebra and $x \in cl_A(xA) \cap cl_A(Ax)$ for all x in A. Some well-known results follow from our results.

2. NOTATION AND PRELIMINARIES

Definitions not explicitly given are taken from Rickart [5].

Let A be a Banach algebra. For any subset E of A, let $cl_A(E)$ denote the closure of E in A and $\ell_A(E)$ (resp. $r_A(E)$) the left (resp. right) annihilator of E in A. Then A is called an annihilator algebra if $\ell_A(A) = r_A(A) = (0)$ and if for every proper closed right ideal I and every proper closed left ideal J $\ell_A(I) \neq (0)$ and $r_A(J) \neq (0)$. If, in addition, $r_A(\ell_A(I)) = I$ and $\ell_A(r_A(J)) = J$, then A is called a dual algebra.

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An idempotent e in a Banach algebra A is said to be minimal if eAe is a division algebra. In case A is semisimple, this is equivalent to saying that Ae (resp. eA) is a minimal left (resp. right) ideal of A.

We say that a Banach algebra A has a right approximate identity if there exists a net $\{u_t\}$ in A such that $x = \lim xu_t$ for all x in A. $\{u_t\}$ is not necessarily bounded. Analogously we define a left approximate identity.

Notation. If A is a Banach algebra which is a dense subalgebra of a Banach algebra B, then we write $\|\cdot\|$ for the norm on A and $|\cdot|$ for the norm on B.

In this paper, all algebras and linear spaces under consideration are over the field C of complex numbers.

3. Banach algebra which is a dense subalgebra in another Banach algebra

In this section, A will be a semisimple Banach algebra which is a dense subalgebra of a semisimple Banach algebra B such that $\|\cdot\|$ majorizes $|\cdot|$ on A. Let E be a subset of A. Then it is clear that $\ell_B(E) = \ell_B(cl_A(E)) = \ell_B(cl_B(E))$ and $r_B(E) = r_B(cl_A(E)) = r_B(cl_B(E))$.

Lemma 3.1. *Let A be a dual algebra. Then for any proper closed right ideal J of A, we have*

$$J = cl_B(J) \cap A = r_B(\ell_B(J)) \cap A.$$

Proof. Since $\ell_A(J) \subset \ell_B(cl_B(J))$, $\ell_B(cl_B(J)) \neq (0)$ and so $cl_B(J) \neq B$. Hence $cl_B(J)$ is a proper closed right ideal of B and so

$$\begin{aligned} J \subset cl_B(J) \cap A &\subset r_B(\ell_B(cl_B(J))) \cap A \\ &= r_B(\ell_B(J)) \cap A \subset r_B(\ell_A(J)) \cap A \\ &= r_A(\ell_A(J)) = J. \end{aligned}$$

Therefore $J = cl_B(J) \cap A = r_B(\ell_B(J)) \cap A$.

Theorem 3.2. *Let A be a dual algebra. Then the following statements are equivalent*:

(1) *B is a dual algebra.*

(2) *For any x in B, $x \in cl_B(xB) \cap cl_B(Bx)$.*

(3) *For any proper closed right (left) ideal R of B, $R = cl_B(R \cap A)$.*

Proof. By [11, p. 79, Theorem 3.2], B is an annihilator algebra and A and B have the same socle S, which is dense in both A and B.

(1) $\Rightarrow$ (2). It follows from [5, p. 105, Corollary (2.8.3)].

(2) $\Rightarrow$ (3). Assume (2). Let R be a proper closed right ideal of B and $x \in R$. Since the socle S is dense in B, we have $x = \lim_n xy_n$ in $|\cdot|$ with y_n in S. Since $xy_n \in S \subset A$, we have $xy_n \in R \cap A$. Hence $x \in cl_B(R \cap A)$ and so $R \subset cl_B(R \cap A)$. Therefore $R = cl_B(R \cap A)$. Similarly, we can show that $R = cl_B(R \cap A)$, if R is a proper closed left ideal of B. Therefore (3) is true.

(3) $\Rightarrow$ (1). Assume (3). Let R be a proper closed right ideal of B. Then by [5, p. 98, Corollary (2.8.7)], R is contained in a maximal modular right ideal M

of B. Therefore by [5, p. 97, Theorem (2.8.5)], $r_B(\ell_B(R)) \subset r_B(\ell_B(M)) = M \neq B$. Hence $r_B(\ell_B(R))$ is a proper closed right ideal of B. Let $J = cl_A(R \cap A)$. Then $cl_B(J) = cl_B(R \cap A) = R$. Since $r_B(\ell_B(J)) = cl_B(r_B(\ell_B(J)) \cap A)$, it follows from Lemma 3.1 that

$$\begin{aligned} r_B(\ell_B(R)) &= r_B(\ell_B(J)) = cl_B(r_B(\ell_B(J)) \cap A) \\ &= cl_B(J) = R. \end{aligned}$$

Similarly, we can show that $\ell_B(r_B(R)) = R$, if R is a closed left ideal of B. Therefore B is a dual algebra. This completes the proof of the theorem.

Corollary 3.3. *Suppose that B has a left approximate identity and a right approximate identity. Then if A is a dual algebra, so is B.*

Proof. For any x in B, it is clear that $x \in cl_B(xB) \cap cl_B(Bx)$. Hence by Theorem 3.2, B is a dual algebra.

Remark 1. Let A be an A^*-algebra which is a dense subalgebra of a B^*-algebra B. It is well known that if A is a dual algebra, so is B, This result also follows from Corollary 3.3, because B has a bounded approximate identity.

Remark 2. A Banach algebra with an unbounded left approximate identity and unbounded right approximate identity may not have a bounded approximate identity (see [2, p. 487, Example 4.2]).

On the other hand, if B is a dual algebra, A may not be a dual algebra. In fact, A may not be an annihilator algebra (for example, see [9, p. 1033] and [10, p. 293]).

The following result is useful in the next section.

Theorem 3.4. *Let B be a dual algebra. Then the following statements are equivalent:*

(1) *A is a dual algebra.*

(2) *For any proper closed right (left) ideal N of A, $N = cl_B(N) \cap A$ and for any proper closed right (left) ideal R of B, $R = cl_B(R \cap A)$.*

Proof. $(1) \Rightarrow (2)$. Assume that A is a dual algebra. Since B is a dual algebra, by Theorem 3.2, $R = cl_B(R \cap A)$. Let N be a proper closed right ideal of A and $x \in cl_B(N) \cap A$. Then there exists a sequence $\{x_n\} \subset N$ such that $x_n \to x$ in $|\cdot|$. Hence for any minimal idempotent e of A, we have $x_n e \to xe$ in $|\cdot|$. Since by [11, p. 78, Lemma 3.1], $\|\cdot\|$ and $|\cdot|$ are equivalent on Ae, $x_n e \to xe$ in $\|\cdot\|$. Since $x_n e \in N$, $xe \in N$, and so $xeA \subset N$. Since e is arbitrary, it follows that $xS_A \subset N$, where S_A is the socle of A and so $cl_A(xA) \subset N$. Therefore, by [5, p. 97, Corollary (2.8.3)], $x \in cl_A(xA) \subset N$. Hence it follows that $cl_B(N) \cap A \subset N$ and so $N = cl_B(N) \cap A$. A similar statement is true for left ideals. Consequently, (2) is true.

$(2) \Rightarrow (1)$. Suppose that (2) is true. Let N be a proper closed right ideal of A. Since $N = cl_B(N) \cap A$, $cl_B(N)$ is a proper closed right ideal of B. Since B is a dual algebra, $\ell_B(N) = \ell_B(cl_B(N)) \neq (0)$. Since $\ell_B(N)$ is a proper closed left ideal of B, by (2), $\ell_B(N) = cl_B(\ell_B(N) \cap A) = cl_B(\ell_A(N))$. In particular, $\ell_A(N) \neq (0)$. Also we have

$$\begin{aligned} N = cl_B(N) \cap A &= r_B(\ell_B(cl_B(N))) \cap A \\ &= r_B(\ell_B(N)) \cap A = r_B(cl_B(\ell_A(N))) \cap A \\ &= r_B(\ell_A(N)) \cap A = r_A(\ell_A(N)). \end{aligned}$$

Similarly, we can show that $J = \ell_A(r_A(J))$ for any closed left ideal J of A. Therefore A is a dual algebra and this completes the proof of the theorem.

Corollary 3.5. *Assume that, for any proper closed right (left) ideal R of B, $R = cl_B(R \cap A)$. Then the following statements are equivalent:*

(1) *A is a dual algebra.*

(2) *B is a dual algebra and, for any proper closed right (left) ideal N of A, $N = cl_B(N) \cap A$.*

Proof. This follows from Theorems 3.2 and 3.4,

The following result is essentially contained in [6, p. 262, Theorem 4.2].

Theorem 3.6. *Let B be a dual algebra. Then the following statements are equivalent:*

(1) *A is a dual algebra.*

(2) *A and B have the same socle S that is dense in A.*

Proof. (1) $\Rightarrow$ (2). Suppose that A is a dual algebra. Then by [11, p. 79, Theorem 3.2], A and B have the same socle that is dense in A.

(2) $\Rightarrow$ (1). This follows from [6, p. 262, Theorem 4.2].

4. Banach algebra which is a dense two-sided ideal in another Banach algebra

In this section, A will be a semisimple Banach algebra which is a dense two-sided ideal of a semisimple Banach algebra B. Then $\|\cdot\|$ majorizes $|\cdot|$ on A, there exists a constant M such that

$$\|ab\| \leq M\|a\|\,|b| \text{ and } \|ba\| \leq M\|a\|\,|b|,$$

for all a in A and b in B, and A and B have the same socle (see [11, p. 78, Lemma 2.1] and [1, p. 3]). (In [1], a slip is made in not assuming that $A = B \cdot A$ in Proposition 3.3 and Theorems 3.4 and 4.2.)

Theorem 4.1. *Suppose that B has a bounded right (resp. left) approximate identity $\{u_t\}$. Then A has a right (resp. left) approximate identity if and only if $x \in cl_A(xA)$ (resp. $x \in cl_A(Ax)$) for all x in A.*

Proof. If A has a right approximate identity, then clearly, $x \in cl_A(xA)$ for all x in A.

Conversely, suppose that $x \in cl_A(xA)$ for all x in A. By [2, p. 486, Lemma 2.1], we can assume that $\{u_t\} \subset A$. We show that $\{u_t\}$ is a right approximate identity of A. Since $\{u_t\}$ is bounded in B, there exists a constant K such that

$|u_t| \leq K$ for all t. Let $x \in A$. Since $x \in cl_A(xA)$, for given $\varepsilon > 0$, there exists $y \in A$, such that $\|x - xy\| < \varepsilon/3MK (< \varepsilon/3)$. Since $\{u_t\}$ is a right approximate identity of B, there exists t_0 such that, for $t > t_0$, $|y - yu_t| < \varepsilon/3M\|x\|$. Therefore,

$$\begin{aligned} \|x - xu_t\| &\leq \|x - xy\| + \|xy - xyu_t\| + \|xyu_t - xu_t\| \\ &\leq \|x - xy\| + M\|x\|\,|y - yu_t| + M\|xy - x\|\,|u_t| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Therefore $\{u_t\}$ is a right approximate identity of A. This completes the proof.

Remark. If A has a bounded right (or left) approximate identity, then $A = B$. In fact, suppose that $\{u_t\}$ is a bounded right approximate identity of A with $\|u_t\| \leq K$, where K is a constant. Then, for each x in A, we have

$$\begin{aligned} \|x\| &\leq \|x - e_t x\| + \|e_t x\| \leq \|x - e_t x\| + M\|e_t\|\,|x| \\ &\leq \|x - e_t x\| + MK|x|. \end{aligned}$$

Since $\|x - e_t x\| \to 0$, it follows that $\|x\| \leq MK|x|$. Therefore $\|\cdot\|$ and $|\cdot|$ are equivalent on A and so $A = B$.

Theorem 4.2. *Suppose that B has a bounded right approximate identity and a bounded left approximate identity. Then the following conditions are equivalent:*

(1) *A is a dual algebra.*

(2) *B is a dual algebra and $x \in cl_A(xA) \cap cl_A(Ax)$ for all x in A.*

Proof. (1) $\Rightarrow$ (2). Assume that A is a dual algebra. Then by [5, p. 105, Corollary (2.8.3)], $x \in cl_A(xA) \cap cl_A(Ax)$ for all x in A. Since B has a bounded right approximate identity and a bounded left approximate identity, by Theorem 3.2, B is a dual algebra.

(2) $\Rightarrow$ (1). Assume that (2) is true. Let $\{u_t\}$ be a bounded right approximate identity of B. Then by Lemma 4.1, we can assume that $\{u_t\}$ is a right approximate identity of A. Let R be a closed right ideal of B and $y \in R$. Since $yu_t \in R \cap A$ and $yu_t \to y$ in $|\cdot|$, it follows that $y \in cl_B(R \cap A)$. Therefore $R \subset cl_B(R \cap A)$, and so $R = cl_B(R \cap A)$. Let N be a closed right ideal of A and $x \in cl_B(N) \cap A$. Write $x = \lim_n x_n$ in $|\cdot|$ with $x_n \in N$. Let $z \in A$. Since $x_n z \in N$ and $\|xz - x_n z\| \leq M|x - x_n|\,\|z\|$, it follows that $xz \in N$; in particular $xu_t \in N$ for all t. Since $xu_t \to x$ in $\|\cdot\|$, it follows that $x \in N$. Therefore $cl_B(N) \cap A \subset N$ and so $N = cl_B(N) \cap A$. A similar statement is true for left ideals. Therefore, by Theorem 3.4, A is a dual algebra. This completes the proof of the theorem.

The following result was proved by Johnson and Lahr (see [3, p. 313, Theorem 2]).

Corollary 4.3. *Let A be an A^*-algebra that is a dense two-sided ideal of a B^*-algebra B. Then A is a dual algebra if and only if B is a dual algebra and A^2 is dense in A.*

Proof. Suppose that A is a dual algebra. By Theorem 4.2, B is a dual algebra. Since the socle of A is dense in A, A^2 is dense in A.

Conversely, suppose that B is a dual algebra and A^2 is dense in A. Then by [3, p. 312, Theorem 1], A has an approximate identity, and so $x \in cl_A(xA) \cap cl_A(Ax)$ for all x in A. Therefore by Theorem 4.2, A is a dual algebra.

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