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ON THE FIXED POINT INDEX OF ITERATES OF PLANAR HOMEOMORPHISMS

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ABSTRACT. If f is an orientation preserving homeomorphism of the plane with an isolated fixed point at the origin 0 and $\text{index}(f, 0) = p$, then $\text{index}(f^n, 0)$ is always well defined provided that $p \neq 1$. In this case, for each $n \neq 0$, $\text{index}(f^n, 0) = \text{index}(f, 0) = p$. If $\text{index}(f, 0) = 1$, then there is an integer p (possibly $p = 1$) such that for those values of n for which $\text{index}(f^n, 0)$ is defined (i.e 0 is an isolated fixed point of f^n), $\text{index}(f^n, 0) = 1$ or $\text{index}(f^n, 0) = p$.

1. INTRODUCTION

If 0 is an isolated fixed point for the continuous map $f: U \to R^m$ where U is an open subset of R^m, then the index of f at 0, $\text{index}(f, 0)$, is the local degree of the mapping $\text{Id} - f$ restricted to an appropriately small open set about 0. If 0 is an isolated fixed point of f^n, then $\text{ind}(f^n, 0)$ is defined for all $n > 0$, where f^n means f composed with itself n times restricted to a small neighborhood of 0. Shub and Sullivan [SS] observed that if $f: U \to R^m$ is C^1 and 0 is an isolated point of f^n for all n, then $\text{index}(f^n, 0)$ is bounded as a function of n. They also provided a simple counterexample if the condition that f be C^1 is weakened to: f is continuous. The purpose of this paper is to show that for homeomorphisms a similar (stronger) result obtains in the case $m = 2$.

Our principal results are as follows: If f is an orientation preserving homeomorphism of the plane with an isolated fixed point at the origin 0 and $\text{index}(f, 0) = p$, then $\text{index}(f^n, 0)$ is always well defined provided that $p \neq 1$. In this case, for each $n \neq 0$, $\text{index}(f^n, 0) = \text{index}(f, 0) = p$. If $\text{index}(f, 0) = 1$, then there is an integer p (possibly $p = 1$) such that for those values of n for which $\text{index}(f^n, 0)$ is defined (i.e 0 is an isolated fixed point of f^n), $\text{index}(f^n, 0) = 1$ or $\text{index}(f^n, 0) = p$. In another paper we shall show that if f is orientation reversing and has an isolated fixed point at the origin, then $\text{index}(f, 0)$ is one of the three values 0, ± 1.

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In §2, we develop the main results in their conceptually simplest form. These are theorems about orientation preserving homeomorphisms of the plane with exactly one fixed point. In §3, we treat the generic (localized) situation of an isolated fixed point.

2.

Definition 1. The flows whose phase portraits are described in Figure 1 are the "canonical flows" of index $p > 1$ as in Figure 1a, $p < 1$ as in Figure 1b and $p = 1$ as in Figure 1c. The canonical flow of index p has $2(1-p)$ hyperbolic

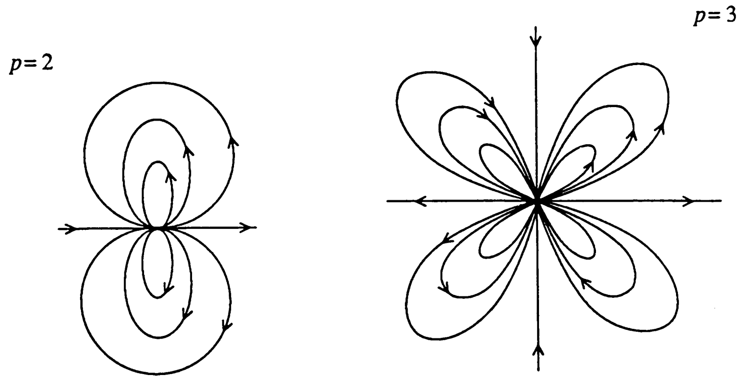

FIGURE 1a.

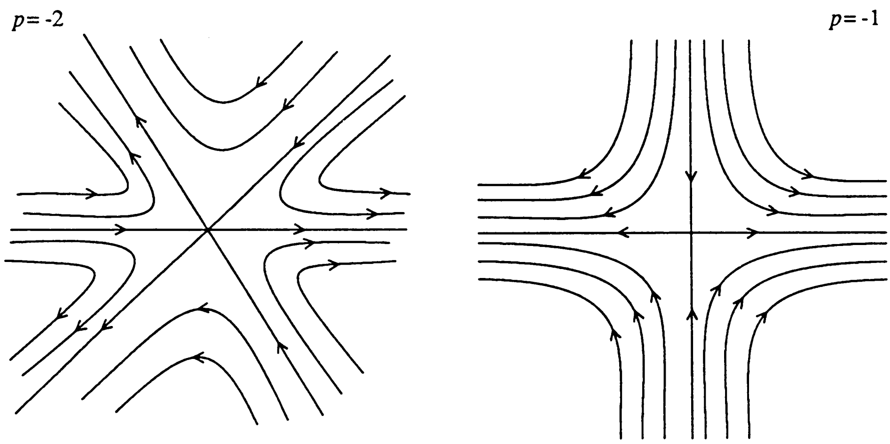

FIGURE 1b.

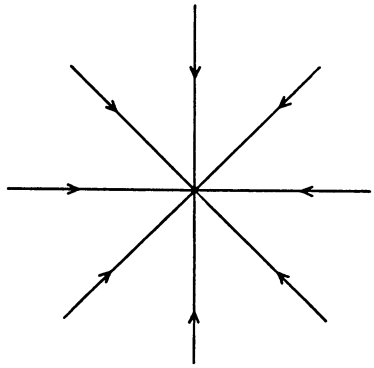

FIGURE 1c.

regions if $p < 1$, and $2(p-1)$ elliptic regions if $p > 1$. *For each* p *we will denote by* h_p *the time one homeomorphism of the canonical flow of* index p. Any conjugate of the homeomorphism h_p has the same phase portrait, and we will refer to any one of these as a canonical homeomorphism of index p. The restriction of the homeomorphism h_p to the unit circle J has image $h(J) = K$ as described in Figure 2 where for each i, the inverse of the arc a_i is the arc b_i. If $p < 1$ then there will be $(1-p)$ arcs a_i. If $p > 1$ there will be $(p-1)$ arcs a_i. Obviously, since h_p is the time one homeomorphism of the canonical flow of index p then the index of h_p^n on J (that is, the index of the nth iterate of h_p restricted to J) = (the index of the time n homeomorphism of the canonical flow of index p) $= p$.

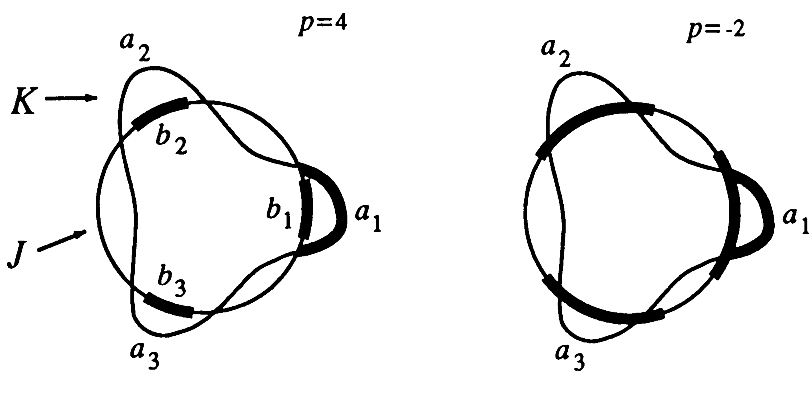

FIGURE 2.

Theorem 1 [S]. *Let f be an orientation preserving homeomorphism of the plane such that the origin 0 is the only fixed point of f, and the index of f at 0 is p. Then there is an isotopy of h to a canonical homeomorphism h_p. Throughout the isotopy the intermediate homeomorphisms have 0 as their only fixed point.*

Definition 2. A homeomorphism f of the plane is "free" provided that whenever D is a topological disk in the plane such that $f(D) \cap D = \Phi$, then for all $i \neq j$, $f^i(D) \cap f^j(D) = \varnothing$. Clearly, the only periodic points of a free homeomorphism are fixed points.

Definition 3. If f is a homeomorphism of an open subset U of the plane into the plane and J is a simple closed curve in U such that f has no fixed points on J then the index of f on J, $\operatorname{index}(f, J)$ is defined as the degree of the map $(f - \mathrm{id})/\|f - \mathrm{id}\|$ from J to the unit circle. Recall that $\operatorname{index}(f, J)$ is invariant under a homotopy $h_t(J_t)$ as long as (1) h_t is fixed point free on J_t throughout, (2) $\operatorname{index}(f, J) = \operatorname{index}(f^{-1}, f(J))$, and (3) if 0 is an isolated fixed point of f then $\operatorname{index}(f, 0) = \operatorname{index}(f, J)$ for any simple closed curve J which surrounds 0 and no other fixed point of f [BN].

Theorem 2 (Brown, [B_2]). *Let f be an orientation preserving homeomorphism of the plane. If f is not free, then there is a simple closed cuve J in the plane such that* $\operatorname{index}(f, J) = 1$.

Corollary. *Let f be an orientation preserving homeomorphism of the plane with a single fixed point 0. Suppose the index of f at 0 is $p \neq 1$. Then f is free, and hence has no periodic points other than the fixed point 0.*

Proof. If J is a simple closed curve such that the origin is not a point of J, then either $\operatorname{index}(f, J) = p$ or $\operatorname{index}(f, J) = 0$, depending upon whether or not J contains the origin in its interior. In either case $\operatorname{index}(f, J) \neq 1$.

Theorem 3. *Let f be an orientation preserving homeomorphism of the plane with the origin 0 as its unique fixed point. Suppose* $\operatorname{index}(f, 0) = p$, *and* $p \neq 1$. *Then for all $n \neq 0$,* $\operatorname{index}(f^n, 0) = \operatorname{index}(f, 0)$.

Proof. According to Theorem 1, f is isotopic to a canonical homeomorphism h_p through an isotopy g_t $(0 \leq t \leq 1,\ g_1 = h_p,\ g_0 = f)$ where g_t has 0 as a unique fixed point. Thus the index of g_t is p for all t, so that each g_t is free, and thus g_t has no periodic points other than the origin. This implies that g_t^n has only the origin as a fixed point. Thus g_t^n is an isotopy from f^n to h_p^n where g_t^n has only 0 as a fixed point. Thus $\operatorname{index}(f^n, 0) = \operatorname{index}(h_p^n, 0) = p$, the last equality coming from the remarks following Definition 1.

Remark. What goes wrong when $p = 1$? Let r denote the rotation of the plane by π, let h be the standard "Anosov" homeomorphism $(h = h_{-1})$: $h(x, y) = (2x, y/2)$, and let $f = hr$. By direct calculation $\operatorname{index}(f, 0) = 1$. On the other hand, $f^2 = h^2$, so $\operatorname{index}(f^2, 0) = -1$. Note that the canonical homeomorphism h_1 maps the unit circle strictly into its interior, so by Theorem 1, f

is isotopic to this map via homeomorphisms whose only fixed point is 0. This can be accomplished easily with f by simply isotoping f, say by f_t along horizontal lines until the f-image of the unit circle lies inside its interior. It is easily seen that f_t has only the origin as a fixed point. Since f^2 and h^2 have different indexes, we must conclude that some points of period 2 are introduced during the isotopy. It follows from the proof of Theorem 3 that any isotopy of f to h_1 must introduce fixed points or points of period two during the isotopy.

Theorem 4. *Suppose f is a homeomorphism of the plane with 0 as its unique fixed point. Suppose that* $\operatorname{index}(f,0) = 1$. *Then either* $\operatorname{index}(f^n,0) = 1$ *for every n for which* $\operatorname{index}(f^n,0)$ *is defined, or there is an integer p such that for each n, either* $\operatorname{index}(f^n,0)$ *is not defined (i.e 0 is not an isolated fixed point of f^n), or* $\operatorname{index}(f^n,0) = 1$ *or* $\operatorname{index}(f^n,0) = p$.

Proof. Obviously, if $\operatorname{index}(f^n,0) = p$, and $\operatorname{index}(f^m,0) = q$, and neither p nor q is equal to 1, then by Theorem 3,

$$p = \operatorname{index}(f^n,0) = \operatorname{ind}(f^{nm},0) = \operatorname{index}(f^m,0) = q.$$

Remark. All situations described in Theorem 4 can occur. If f is the composition of the canonical index p homeomorphism $(p > 2)$ with a rotation by $2\pi/(p-1)$ then f has index 1 for $1 \le n \le p-1$, and $\operatorname{index}(f^n,0) = \operatorname{index}(h_p)^n = \operatorname{index}(h_p,0) = p$. A similar construction obtains for $p < 0$. (I do not know of examples where $p = 0$ or 2.) On the other hand, if f is a rotation by an irrational multiple of 2π then $\operatorname{index}(f^n,0) = 1$ for all n. Finally if f is a "rational" rotation, then for certain values of f and n, 0 is not an isolated fixed point of f^n.

3.

In this section we study orientation preserving homeomorphisms of the plane in which the origin is an isolated fixed point. We do not assume that the origin is isolated as a periodic point. Nevertheless, we obtain results similar to those of §1. It will be necessary at times, in order to get the slightly stronger results that are needed, to refer to the methods of [S], [B_1], and [B_2]. Theorem 5, for example, is a slightly refined version of Theorem 5.7 of [B_2]. The reader who wishes to verify this should retrace the argument in [B_1] or [B_2].

Theorem 5. *Let f be a homeomorphism of an open set U of the plane into the plane. Suppose x is a point of U such that* (1) $f(x) \neq x$, (2) *for some* $n > 1$, $f^i(x)$ *is in U for* $1 \le i \le n$, *and* (3) $f^n(x) = x$. *Suppose there exists a connected neighborhood W of* $\{x\} \cup \{f(x)\}$ *such that* $f^i(W)$ *is a subset of U for* $1 \le i \le n$, *and W contains no fixed points of f. Then there is a simple closed curve J in U such that* $\operatorname{index}(f,J) = 1$.

Theorem 6. *Let f be an orientation preserving homeomorphism of the plane such that 0 is an isolated fixed point of f and* $\operatorname{index}(f,0) = p \neq 1$. *Then for each $n > 0$, 0 is an isolated fixed point of f^n.*

Proof. Let $n > 0$ be given, and let U be a neighborhood of 0 containing no fixed points of f other than 0. Then the index of f on any simple closed curve in U is either p or zero. Let V be a connected neighborhood of 0 such that $f^i(V)$ is a subset of U for each i, $1 \leq i \leq n+1$. Let x be a point of V other than 0. If $f^i(x) = x$ for $1 < i \leq n$, then by Theorem 5, with $W = V \cup f(V)$, U contains a simple closed curve on which the index of f is 1. This contradicts the assumption that $p \neq 1$.

Remark. One must choose V *after* n is given, since 0 might not be isolated as a *periodic point.* Smale's horseshoe provides such an example.

Theorem 7 [S]. *Let f be a homeomorphism of the plane such that 0 is an isolated fixed point of f and* $\text{index}(f, 0) = p$. *Let U be a neighborhood of* 0. *Then there is a neighborhood V of the origin and an isotopy h_t connecting f to a homeomorphism g such that* (1) $g|v$ *is canonical of* index p, (2) *for all t, $0 \leq t \leq 1$, $h_t|(R^2 - U) = f|(R^2 - U)$, and* (3) $h_t|U$ *has* 0 *as a unique fixed point.*

Remark. This is a modified version of the main theorem of [S]. (In paragraph 4, p. 239 apply the Alexander isotopy on one side only.) As a consequence of Theorems 5–7 and the methods of §1 we have Theorem 8.

Theorem 8. *The conclusions of Theorem* 4 *hold if we assume only that f has* 0 *as an isolated fixed point instead of a unique fixed point.*

References

[B 1] M. Brown, *A new proof of Brouwer's Lemma on translation arcs*, Houston J. Math. **10** (1984), 35–41.

[B 2] ——, *Homeomorphisms of two-dimensional manifolds*, Houston J. Math. **11** (1985), 455–469.

[BN] M. Brown and W. Neumann, *Proof of the Birkhoff-Poincaré fixed point theorem*, Michigan Math. J. **24** (1977), 21–31.

[S] Bruno V. Schmitt, *L'espace des homeomorphismes du plan qui admettant un seul point fixe d'indice donne est connexe par arcs*, Topology **18** (1979), 235–240.

[SS] M. Shub and D. Sullivan, *A remark on the Lefschetz fixed point formula for differentiable maps*, Topology **13** (1974), 189–191.

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