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RIBBON CONCORDANCE DOES NOT IMPLY A DEGREE ONE MAP

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ABSTRACT. We give an example of classical knots K_0, K_1 such that (1) K_1 is ribbon concordant to K_0, (2) there are no degree one maps from the exterior of K_1 in S^3 to that of K_0.

1.

Throughout this note let K_0 and K_1 denote classical knots, A_i denote the Alexander module of K_i, and X_i the exterior of K_i in S^3 for $i = 0, 1$. Let $\Lambda = Z[t, t^{-1}]$.

In [2] Gordon introduced the notion of ribbon concordance. We say K_1 is *ribbon concordant* to K_0 (and write $K_1 \geq K_0$) if there is an annulus C in $S^3 \times I$ such that $C \cap S^3 \times \{i\} = K_i$, $i = 0, 1$, and the restriction to C of the projection $S^3 \times I \to I$ is a Morse function with no local maxima. Gordon asked:

Question 1 ([2], 6.4). *Let* $v(K_i)$ *denote the Gromov norm of* X_i. *Does* $K_1 \geq K_0$ *imply* $v(K_1) \geq v(K_0)$?

If there were a degree one map from X_1 to X_0, then an affirmative answer to the question would follow from the property of the Gromov norm. Such a degree one map would also imply that A_0 be a quotient of A_1. This is observed by Gilmer [1], and he asks:

Question 2 ([1], 4.6). *Does* $K_1 \geq K_0$ *imply that there is a* Λ*-epimorphism from* A_1 *to* A_0?

In this paper we give a negative answer to this question.

Proposition 1. *There are* K_0 *and* K_1 *such that*

(1) $K_1 \geq K_0$,

(2) *there are no* Λ*-epimorphisms from* A_1 *to* A_0.

In particular, there are no degree one maps from X_1 *to* X_0.

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In fact Gilmer's question is posed in an algebraically generalized form, but the proposition still gives a "no" answer. However, Question 1 remains open.

I would like to thank Professor Gordon for suggesting this problem to me.

2.

Let K_0 be a knot with a Seifert matrix V_0 below. Let K_1

$$V_0 = \begin{pmatrix} 13 & 1 \\ 0 & 1 \end{pmatrix} \qquad V_1 = \begin{pmatrix} 13 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 1 & 1 \end{pmatrix}$$

be a knot with a Seifert matrix V_1 and $K_1 \geq K_0$. The existence of K_1 is guaranteed by [1, Theorem (1.3)]. Simplify the presentation matrix $tV_1 - V_1^T$ of A_1 as follows.

$$tV_1 - V_1^T = \begin{pmatrix} 13(t-1) & t & 0 & 3(t-1) \\ -1 & t-1 & 0 & 0 \\ 0 & 0 & 0 & 2t-1 \\ 3(t-1) & 0 & t-2 & t-1 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 13(t-1)^2 + t & 0 & 3(t-1) \\ -1 & t-1 & 0 & 0 \\ 0 & 0 & 0 & 2t-1 \\ 3(t-1) & 0 & t-2 & t-1 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 13(t-1)^2 + t & 0 & * \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2t-1 \\ 3(t-1) & 3(t-1)^2 & t-2 & ** \end{pmatrix}$$

$$\to \begin{pmatrix} 13t^2 - 25t + 13 & 0 & * \\ 0 & 0 & 2t-1 \\ 3(t-1)^2 & t-2 & ** \end{pmatrix}$$

$$\to \begin{pmatrix} 13t^2 - 25t + 13 & 0 & * \\ 0 & 0 & 2t-1 \\ 3 & t-2 & ** \end{pmatrix} \equiv M .$$

Therefore A_1 is generated by three elements, say α, β and γ, subject to the relations

$$M \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0 .$$

It is easy to see that A_0 is the cyclic Λ-module of order $13t^2 - 25t + 13$. Note that A_0 contains $(t-1)^{-1}$.

Lemma 1. *Suppose that there is a Λ-epimorphism $f\colon A_1 \to A_0$. Then there are u and v in A_0 such that*

$$3u + (x-1)v = 0 \qquad \textit{and } (u, v) = (1), \qquad \textit{where } x = (t-1)^{-1}.$$

Proof. Then $f(\alpha)$, $f(\beta)$ and $f(\gamma)$ generate A_0 and satisfy the following equations.

$$(13t^2 - 25t + 13)f(\alpha) + *f(\gamma) = 0, \tag{1}$$

$$(2t-1)f(\gamma) = 0, \tag{2}$$

$$3f(\alpha) + (t-2)f(\beta) + **f(\gamma) = 0. \tag{3}$$

Since $13t^2 - 25t + 13$ is irreducible in Λ, (2) shows $f(\gamma) = 0$. It follows from (3) that

$$3f(\alpha) + (t-2)f(\beta) = 0.$$

Multiplying by $x = (t-1)^{-1}$, we get

$$3xf(\alpha) + (1-x)f(\beta) = 0.$$

The desired equation follows by setting $u = xf(\alpha)$ and $v = -f(\beta)$. □

We shall show that A_0 does not have the elements u and v as in Lemma 1. Express A_0 in terms of x where $x = (t-1)^{-1}$; then A_0 is the cyclic $Z[1 + x^{-1}, (1+x^{-1})^{-1}]$ module of order $1 + x^{-1} + 13x^{-2}$. Since $x \in A_0$ and $(1+x^{-1})^{-1} = x(x+1)^{-1}$. We obtain:

$$A_0 \cong Z[x, x^{-1}, (1+x)^{-1}]/(x^2 + x + 13).$$

Let $D = Z[x]/(x^2+x+13)$ and S be the multiplicative set of D generated by x and $x+1$. It follows that $A_0 \cong S^{-1}D$. Since D is $Z[(-1+\sqrt{-51})/2]$ which is the ring of integers in $Q(\sqrt{-51})$, in particular a Dedekind domain. In fact A_0 is also a Dedekind domain. The following algebraic lemmas establish Proposition 1.

Lemma 2. *If an ideal P is prime and nonprincipal in D, then so is $PS^{-1}D$ in $S^{-1}D$.*

Lemma 3. *In D the following hold:*

(1) $(3) = (3, x-1)^2$,
(2) $(x-1) = (3, x-1)(5, x-1)$,
(3) $(3, x-1)$ *and* $(5, x-1)$ *are prime but nonprincipal ideals in D, (and hence also in $S^{-1}D$ by Lemma 2).*

Proof of Proposition 1. If there is an epimorphism from A_1 to A_0, by Lemma 1 there are u and v in $S^{-1}D$ such that $3u + (x-1)v = 0$ and $(u, v) = (1)$. We have the ideal equation

$$(3)(u) = (x-1)(v).$$

By Lemma 3 we obtain

$$(3, x-1)^2(u) = (3, x-1)(5, x-1)(v) \quad \text{in } S^{-1}D.$$

Since an ideal in the Dedekind domain $S^{-1}D$ has a unique prime ideal decomposition, it follows that:

$$(u) = (5, x-1)Q \text{ and } (v) = (3, x-1)Q \text{ for some ideal } Q.$$

The ideal (u) is principal, but $(5, x-1)$ is not. It follows that $Q \neq (1)$. Thus $(u, v) = Q \neq (1)$, a contradiction to Lemma 1. The proof is completed. □

Proof of Lemma 2. If $P \cap S \neq \emptyset$, P contains a prime element x or $x+1$. Thus P is a principal ideal, a contradiction. It follows $P \cap S = \emptyset$. Then $PS^{-1}D$ is prime. If $PS^{-1}D$ is principal, there is a $b \in P$ such that $PS^{-1}D = bS^{-1}D$. Among all such b take one such that bD is maximal. This is possible because D is Noetherian. Then b is not divisible by x or $x+1$. For an arbitrary $p \in P$ there are $s_1, s_2 \in S$ and $d \in D$ such that $p/s_1 = bd/s_2$. Thus $s_2 p = by$ for some $y \in D$. Let $s_2 = p_1 \cdots p_r$ with $p_i = x$ or $x+1$; then $p_i \nmid b$, so that $p_i | y$. An induction on r shows that y is divisible by s_2, so $p \in bD$. It follows that $P = bD$. This contradicts the assumption that P is not principal. Thus $PS^{-1}D$ is not principal. □

Proof of Lemma 3. Since $D/(3, x-1) = Z_3[x]/(x-1) \cong Z_3$, a domain, $(3, x-1)$ is a prime ideal of norm 3. On the other hand we see that $(3, x-1)^2 \subset (3)$, for $(x-1)^2 = -3(x+4)$ in D. Since the norm of $(3, x-1)^2$ is 9, it follows $(3, x-1)^2 = (3)$.

If $(3, x-1)$ were principal, it could be written as

$$\left(a + b\frac{-1+\sqrt{-51}}{2}\right) \quad \text{where } a, b \in Z.$$

Then the norm of $(3, x-1)$ is $((2a-b)^2 + 51b^2)/4$, which must be 3. However, there are no integral solutions of

$$(2a-b)^2 + 51b^2 = 12.$$

Thus $(3, x-1)$ is not principal. By the similar arguments we can prove the conclusions about $(x-1)$ and $(5, x-1)$. □

References

1. P. Gilmer, *Ribbon concordance and a partial order on S-equivalence classes*, Topology Appl. **18** (1984), 121–144.

2. C. Gordon, *Ribbon concordance of knots in the 3-sphere*, Math. Ann. **257** (1981), 157–170.

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