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ON THE MORAVA K-THEORIES OF $SO(2n+1)$

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ABSTRACT. In this paper we compute the additive structure of the Morava K-theories of $SO(2n+1)$. We also obtain partial information about the bialgebra structures, and indicate how one may compute the effect of the stable BP-operations and the Milnor primitives.

1. STATEMENT OF RESULTS

In this paper we bring the program started in [Ra1] to a partial close by determining the additive structure of the Morava K-theories of $SO(2n+1)$. The theorem we prove contains incomplete information concerning the bialgebra structure and the effect of the stable BP-operations and the Milnor primitives. This paper is a sequel to [Ra2], to which the reader is referred for a proper introduction.

Throughout this paper BP will refer to the 2-local theory. For background information on BP and related topics, see [Wi].

It is well known that $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \ldots]$ where the degree of v_i is $2(2^i - 1)$ (see, for example, [Qu]). Let $l > 0$. Using the Sullivan-Baas technique ([Su], [Bs]), we can kill $2, v_1, \ldots, v_{l-1}$. The resulting spectrum is $P(l)$, a BP-module spectrum with $P(l)_* = \mathbb{Z}/2[v_l, v_{l+1}, \ldots]$. (This and the next few statements are due to Jack Morava. See [JW] for a source in print.) It is consistent to define $P(0) = BP$ and $P(\infty) = H\mathbb{Z}/2$. Killing $\{v_i \mid i \neq l\}$ gives $k(l)$, the connective Morava K-theory. Inverting v_l gives $B(l) = v_l^{-1}P(l)$ and the (periodic) Morava K-theory $K(l) = v_l^{-1}k(l)$.

If $0 < l < \infty$, there are two products on $P(l)$ which make it into a BP-algebra theory [Wu]. Select one and give $B(l)$, $k(l)$ and $K(l)$ the compatible products. For any space X, $B(l)_*(X)$ is free as a $B(l)_*$-module and $K(l)_*(X) \cong K(l)_* \otimes B(l)_*(X)$ [JW]. Note that if X is an H-space, then $B(l)_*(X)$ is a bialgebra. But it need not be a Hopf algebra, as $B(l)$ is not commutative.

Our calculations are based on the bar spectral sequence [RS]. In fact, we will determine the E^∞-term of the bar spectral sequence (Bss) converging to

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$B(l)_*(SO(2n+1))$. Clearly there are no additive extension problems. The multiplicative extension problem is complicated by the fact that $B(l)_*(SO(2n+1))$ need not be a Hopf algebra. We will not even attempt to solve the extension problem. We do, however, obtain a good hold on the algebra generators.

Petrie's results [Pe, §I.3] give $v_1^{-1} BP_*(Spin(q))$. One can easily deduce $B(1)_*(SO(q))$ from this. Our result can be seen as a generalization. In fact the determination of the E^2-terms is almost identical. However, the Bss collapses at E^2 in Petrie's case. But in general, there are further non-zero differentials.

Throughout this paper we will fix the values of l and n. We will also assume that $n \geq 2^l$. Define $m = [n/2]$, $m' = [(n-1)/2]$ and $k(i) = l - [\log_2(2i+1)]$ for $0 \leq i < 2^{l-1}$. We denote the truncated algebra of divided powers on t, of height k, by $\Gamma_k(x)$; this is the dual of the truncated polynomial algebra $P(x)/(x^{2^k})$. The basis dual to $\{x^i\}$ is denoted by $\{\gamma_i(t)\}$, the "divided powers" of t. The full divided power algebra, $\bigcup_k \Gamma_k(t)$, is denoted by $\Gamma(t)$.

Our main result, which will be proved in §3, is

Theorem 1. *There is an increasing filtration on $B(l)_*(SO(2n+1))$ that respects the bialgebra structure. The bigraded algebra associated to this filtration is*

$$\bigotimes_{i=2^{l-1}}^{\infty} E(\overline{\beta}_{2i}) \otimes \bigotimes_{i=0}^{2^{l-1}-1} \Gamma_{k(i)}(\gamma_i) \qquad \textit{if } n = \infty;$$

$$\bigotimes_{i=2^{l-1}}^{m'-2^{l-1}} E(\overline{\beta}_{2i}) \otimes \bigotimes_{i=2m'-2^l+2}^{n-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=m}^{n-1} E(\overline{\alpha}_{2i+1}) \otimes \bigotimes_{i=0}^{2^{l-1}-1} \Gamma_{k(i)}(\gamma_i)$$

$$\textit{if } n \geq 2^{l+1};$$

$$\left(\bigotimes_{i=0}^{2^{l-1}-1} \Gamma_{k(i)+1}(\overline{\beta}_i) \otimes \bigotimes_{i=2^{l-1}}^{n-1} E(\overline{\beta}_i) \right) \Big/ (\overline{\beta}_i \mid 0 \leq i \leq n - 2^l) \otimes \bigotimes_{i=2^l-1}^{n-1} E(\overline{\alpha}_{2i+1})$$

$$\textit{if } 2^l \leq n < 2^{l+1}$$

where the bidegress of $\overline{\beta}_i$, $\overline{\alpha}_j$ and γ_k are $(1, 2i)$, $(1, 2j)$ and $(2, 4k)$ respectively.

Remark. If $n < 2^l$, [Ra2, Theorem 1.1] gives the corresponding result.

2. Preliminaries

This section is a "summary of a summary" whose main purpose is to set up the notation. The reader is referred to [Ra2, §§2, 3 and 4] for a more leisurly account and further references.

First we recall the basic facts concerning the bar spectral sequence: Let h be a ring spectrum and M a topological monoid such that the product pairing $\otimes_1^n h_*(M) \to h_*(\times_1^n M)$ is an isomorphism of algebras for all n. Then there is a (first and fourth quadrant) spectral sequence with E^2-term $Tor_{**}^{h_*(M)}(h_*, h_*)$. It converges to $h_*(BM)$, where BM is the classifying space of M, in the following sense: There is an increasing filtration $\{F_{p,*-p} \mid p \geq 0\}$ of $h_*(BM)$ such that

$$E^\infty_{pq} = F_{pq}/F_{p-1,q+1} \text{ and } h_n(BM) = \bigcup_{p=0}^{\infty} F_{p,n-p}\ ;$$

but, if h is not connective, infinitely many of the inclusions $F_{p-1,q+1} \subset F_{pq}$ may be strict. Note that $E^r_{0*} = h_*$ and that E^2_{1*} consists of permanent cycles. The homology suspension factors as

$$\widetilde{h}_*(M) \to Tor^{h_*(M)}_{1*}(h_*, h_*) = E^2_{1*} \to E^\infty_{1*} \to \widetilde{h}_*(BM)\ ,$$

where the first map is the "algebraic homology suspension" (see [Ca] for a definition).

Note: In our indexing, the filtration degree of E^r_{pq} is p; this differs from the indexing used in [Pe].

We denote the Bss with $M = \Omega SO(2n+1)$ and h a BP-algebra spectrum by $E^*_{**}(SO(2n+1), h)$. Note that $BM \simeq SO(2n+1)$. By [Ra2, Theorem 3.1.(3)], $E^*_{**}(SO(2n+1), B(l))$ is a spectral sequence of bicommutative Hopf algebras (even though $B(l)$ is not commutative). It turns out that $E^\infty_{p*}(SO(2n+1), B(l))$ is trivial for large p. So the Bss converges to $B(l)_*(SO(2n+1))$ in the strongest possible sense.

Strictly speaking, the ranges of the indices as given in the rest of this paper apply only to the case of finite n. However, the modifications needed for the case $n = \infty$ should be clear.

Recall that if h is complex oriented, then there is a formal group law $F(x, y) \in h_*[[x, y]]$ [Qu]. The [2]-series is $[2](x) = F(x, x)$. The $[-1]$-series is the formal group inverse, uniquely defined by $F(x, [-1](x)) = 0$. As the coefficients of these series exist already in MU_*, we may suppress h from the notation.

Let $Q_n = SO(n+2)/(SO(2)\times SO(n))$ be the generating variety for the homology of $\Omega_0 SO(n+2)$; *i.e.* there is a map $Q_n \to \Omega_0 SO(n+2)$ such that $H_*(Q_n, \mathbb{Z})$ maps monomorphically into $H_*(\Omega_0 SO(n+2), \mathbb{Z})$ and the image of the former generates the latter as an algebra [Bt, §9]. The direct limit of Q_n is $\mathbb{C}P^\infty$. Let x be the Conner-Floyd Chern class of the canonical complex line bundle over Q_n. Then $MU\mathbb{Q}^*(Q_{2n-1}) = MU\mathbb{Q}^*[x]/(x^{2n})$. Let $\beta'_0, \beta_1, \ldots, \beta_{2n-1}$ be the basis of $MU\mathbb{Q}_*(Q_{2n-1})$ dual to $\{x^i \mid 0 \leq i < 2n\}$. We identify the elements of $MU_*(Q_{2n-1})$ with their images in $MU_*(\Omega SO(2n+1))$. Let β_0 be the generator of $\widetilde{MU}_0(\Omega SO(n))$ such that $\beta_0^2 = 2\beta_0$. For $0 \leq j < 2n$, put $\alpha_j = \sum_{i=0}^{j} c_{j-i}\beta_i$ where c_i is the coefficient of x^{i+1} in $[2](x)$.

If $j < 2n$, then $\alpha_j \in MU_*(\Omega SO(2n+1))$ [Ra1, Theorem 2.3]. The latter is generated by β_i, $0 \leq i < n$ and α_{2j+1}, $m \leq j < n$, subject to the relations

$\beta_0^2 = 2\beta_0$ and

$$(1) \qquad \left(1 + \sum_{i=1}^{m'} \beta_i t^i\right)\left(1 + \sum_{i=1}^{m'} \beta_i([-1]t)^i\right) = 1 \pmod{t^{2m'}}$$

([Ra1, Theorem 2.3]; see also [Pe, §I.2]). This implies relations of the form

$$\beta_k^2 = \sum_{0<i} b_{ki}\beta_i + \sum_{0<i<j} d_{kij}\beta_i\beta_j \qquad \text{if } 1 \le k \le m'.$$

Let $\varepsilon_0 = 2\beta_0$ and $\varepsilon_k = \sum_i b_{ki}\beta_i$ for $1 \le k \le m'$. We will use the same symbols to denote the images of these elements in $h_*(\Omega SO(2n+1))$, where h is any BP-algebra spectrum.

We will calculate the E^2-term of the Bar spectral sequence as the homology of the "Petrie complex". This complex is based on a resolution used in [Pe, §I.3]. The algebra underlying it is

$$M = \bigotimes_{i=0}^{n-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=m}^{n-1} E(\overline{\alpha}_{2i+1}) \otimes \bigotimes_{i=0}^{m'} \Gamma(\gamma_i)$$

where the bidegrees of $\overline{\alpha}_i$, $\overline{\beta}_j$ and γ_k are $(1, 2i)$, $(1, 2j)$ and $(2, 4k)$ respectively. If $x = \sum_{i=0}^{n-1} a_i\beta_i + \sum_{j=m}^{n-1} b_j\alpha_{2j+1}$, where $a_i, b_j \in B(l)_*$, let $\overline{x} = \sum a_i\overline{\beta}_i + \sum b_j\overline{\alpha}_{2j+1}$. We define a derivation d of bidegree $(-1, 0)$ on M by $d(\overline{\beta}_i) = 0 = d(\overline{\alpha}_j)$ and $d(\gamma_{ki}) = \overline{\varepsilon}_k\gamma_{k,i-1}$; here γ_{ki} is the ith divided power of γ_k. Then (M, d) is the Petrie complex. It is consistent with our notation to denote the image of x under the suspension $B(l)_*(\Omega SO(2n+1)) \to E^2_{1*}(SO(2n+1), B(l))$ by $\overline{x}$.

Fix a gound ring of characteristic 2. Given elements β and α of bidegrees (a, b) and $(c, 2^k(a+b)-c-1)$ respectively, define a spectral sequence of Hopf algebras by

$$E^r_{**}(\beta, \alpha) = \begin{cases} E(\alpha) \otimes \Gamma(\beta) & r \le 2^k a - c \\ \Gamma_k(\beta) & r > 2^k a - c \end{cases}$$

$$d^r(\gamma_j(\beta)) = \begin{cases} 0 & r < 2^k a - c \text{ or } j < 2^k \\ \alpha\gamma_{j-2^k}(\beta) & r = 2^k a - c \text{ and } j \ge 2^k. \end{cases}$$

For $0 \le i < 2^{l-1}$, let $j(i) = 2^{k(i)}(2i+1) - 1$ and $j'(i) = j(i) - 2^l + 1$. Note that $j(i)$ is odd and that $2^l - 1 \le j(i) \le 2^{l+1} - 3$. The differentials of the Bss will follow from the following corollary of [Ra2, Theorem 1.1]:

Proposition 2. *As a spectral sequence of Hopf algebras,*

$$E^*_{**}(SO(2^{l+1}-1), B(l)) = \bigotimes_{i=2^{l-1}}^{2^l-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=0}^{2^{l-1}-1} E^*_{**}(\overline{\beta}_i, \overline{\alpha}_{j(i)}).$$

If $k < 2^{l-1}$, *then* γ_{ki} *in the Petrie complex represents* $\gamma_{2i}(\overline{\beta}_k)$.

3. Proofs

Lemma 1. *If* $k \le \min(2^{l-1}-1, m')$ *then* $\varepsilon_k = 0$ *in* $B(l)_*(\Omega SO(2n+1))$. *If* $r \le m' - 2^{l-1}$, *then*

$$\{\beta_i \mid 0 \le i \le 2r+1\} \text{ and } \{\beta_{2i} \mid 0 \le i \le r\} \cup \{\varepsilon_i \mid 2^{l-1} \le i \le 2^{l-1}+r\}$$

are bases for the same subcomplex of $B(l)_*(\Omega SO(2n+1))$.

Proof. Define $\nu_{ij} \in B(l)_*$ by $([-1]t)^i = \sum_j \nu_{ij} t^{i+j}$. Let $k \le m'$. A straightforward calculation starting from 2.1 shows that, in MU_*,

$$(-1)^{k+1} b_{ki} = \begin{cases} 2 & i = 2k \\ \nu_{i,2k-i} + \sum_{i/2 \le j < k} \nu_{j,2k-2j} b_{ji} & 1 \le i < 2k. \end{cases}$$

Now $[-1]t = t + v_l t^{2^l} \bmod (t^{2^{l+1}})$ in $B(l)_*$ (see [Mr, p. 83]). So

$$\nu_{ij} = \begin{cases} 0 & 0 < j < 2^{l+1}-2 \text{ and } j \ne 2^l - 1 \\ v_l & j = 2^l - 1 \text{ and } i \text{ is odd.} \end{cases}$$

Using these two equations in the definition of ε_k, we get

$$\varepsilon_k = \begin{cases} 0 & k < 2^{l-1} \\ v_l \beta_{2k-2^l+1} + \sum_{j=1}^{2k-2^{l+1}+2} b_{kj}\beta_j & 2^{l-1} \le k \le m'. \end{cases}$$

The lemma follows as v_l is invertible in $B(l)_*$.

Remark. This also follows from [Pe, Lemma I.2.1]. However the calculation above is useful in computing the stable BP-operations.

Proposition 2. $E^*_{**}(SO(2n+1), B(l))$ *is isomorphic to*

$$\bigotimes_{i=2^{l-1}}^{\infty} E(\overline{\beta}_{2i}) \otimes \bigotimes_{i=0}^{2^l-1} E^*_{**}(\gamma_i, v_l\overline{\beta}_{j'(i)}) \qquad \text{if } n = \infty;$$

$$\bigotimes_{i=2^{l-1}}^{m'-2^{l-1}} E(\overline{\beta}_{2i}) \otimes \bigotimes_{i=2m'-2^l+2}^{n-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=m}^{n-1} E(\overline{\alpha}_{2i+1}) \otimes \bigotimes_{i=0}^{2^{l-1}-1} E^*_{**}(\gamma_i, v_l\overline{\beta}_{j'(i)})$$

$$\text{if } n \ge 2^{l+1};$$

$$\bigotimes_{i=n-2^l+1}^{n-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=2^l-1}^{n-1} E(\overline{\alpha}_{2i+1}) \otimes \bigotimes_{j(i)<n} E^*_{**}(\gamma_i, v_l\overline{\beta}_{j'(i)}) \otimes \bigotimes_{n \le j(i)} E^*_{**}(\gamma_i, \overline{\alpha}_{j(i)})$$

$$\text{if } 2^l \le n < 2^{l+1}$$

as a spectral sequence of algebras.

Proof. Excepting the need for more vigilent bookkeeping of indices, the proofs of the last two cases are similar to the first. So the full details will be given only for the case $n = \infty$.

It follows from Lemma 1 that the Petrie complex can be written as

$$\bigotimes_{i=0}^{\infty} E(\overline{\beta}_{2i}) \otimes \bigotimes_{i=0}^{2^{l-1}-1} \Gamma(\gamma_i) \otimes \bigotimes_{i=2^{l-1}}^{\infty} (\Gamma(\gamma_i) \otimes E(\overline{\varepsilon}_i)) \quad .$$

The differential is trivial on the first two factors. The last factor is acyclic because $d(\gamma_{ki}) = \overline{\varepsilon}_k \gamma_{k,i-1}$. Combining all this with the fact that $\{2i \mid 0 \leq i < 2^{l-1}\} = \{j'(i) \mid 0 \leq i < 2^{l-1}\}$, we see that

$$E^2_{**}(SO, B(l)) = \bigotimes_{i=2^{l-1}}^{\infty} E(\overline{\beta}_{2i}) \otimes \bigotimes_{i=0}^{2^{l-1}-1} \left(\Gamma(\gamma_i) \otimes E(v_l \overline{\beta}_{j'(i)})\right) \ .$$

This is the E^2-term for the right hand side.

It follows from [Ra2, Lemma 4.3] that

$$E^*_{**}(SO(2^{l+1}-1), B(l)) \longrightarrow E^*_{**}(SO, B(l))$$

sends $\overline{\alpha}_{j(i)}$ to $v_l \overline{\beta}_{j'(i)}$. Combining this with Proposition 2.2, we see that the differentials are as claimed.

In the case $n \geq 2^{l+1}$, the only modification needed is in determining the E^2-term: ε_i is relevant only when $i \leq m'$. Thus the Petrie complex can be written as

$$\bigotimes_{i=0}^{m'-2^{l-1}} E(\overline{\beta}_{2i}) \otimes \bigotimes_{i=2m'-2^l+2}^{n-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=m}^{n-1} E(\overline{\alpha}_{2i+1}) \otimes \bigotimes_{i=0}^{2^{l-1}-1} \Gamma(\gamma_i) \otimes \bigotimes_{i=2^{l-1}}^{m'} (\Gamma(\gamma_i) \otimes E(\overline{\varepsilon}_i)) \ .$$

The last factor is acyclic and the differential is trivial on the others. It follows that the E^2-term is

$$\bigotimes_{i=2^{l-1}}^{m'-2^{l-1}} E(\overline{\beta}_{2i}) \otimes \bigotimes_{i=2m'-2^l+2}^{n-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=m}^{n-1} E(\overline{\alpha}_{2i+1}) \otimes \bigotimes_{i=0}^{2^{l-1}-1} \left(\Gamma(\gamma_i) \otimes E(v_l \overline{\beta}_{j'(i)})\right) \ .$$

Finally consider the case $2^l \leq n < 2^{l+1}$. The Petrie complex is as in the previous case. The rest differs as $\overline{\alpha}_{j(i)} \mapsto v_l \overline{\beta}_{j'(i)}$ iff $j(i) < n$. Note that $\overline{\alpha}_{j(i)}$ is one of the listed exterior generators of $E^2_{**}(SO(2n+1), B(l))$ if $j(i) \geq n$. Using the facts that $\{j'(i) \mid j(i) < n\} = \{2j \mid j \leq m - 2^{l-1}\}$, and $\{j(i) \mid j(i) \geq n\} = \{2j+1 \mid m \leq j < 2^l - 1\}$, we can show that $E^2_{**}(SO(2n+1), B(l))$ is isomorphic to

$$\bigotimes_{i=n-2^l+1}^{n-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=2^l-1}^{n-1} E(\overline{\alpha}_{2i+1}) \otimes \bigotimes_{j(i)<n} \left(\Gamma(\gamma_i) \otimes E(v_l \overline{\beta}_{j'(i)})\right) \otimes \bigotimes_{n \le j(i)} \left(\Gamma(\gamma_i) \otimes E(\overline{\alpha}_{j(i)})\right).$$

The differentials are determined as before. The reader should be able to complete the proof.

Lemma 3. *Suppose that* $i < 2^{l-1}$ *and that* $j < 2^{k(i)+1}$. *Then, under*

$$\bigotimes_{i=0}^{2^{l-1}-1} \Gamma_{k(i)+1}(\overline{\beta}_i) \otimes \bigotimes_{i=2^{l-1}}^{2^l-1} E(\overline{\beta}_i) = E_{**}^{\infty}(SO(2^{l+1}-1), B(l)) \xrightarrow{\text{incl.}_*} E_{**}^{\infty}(SO(2n+1), B(l)),$$

$$\gamma_j(\overline{\beta}_i) \mapsto \begin{cases} \gamma_{i,j/2} & j \text{ even}; \\ \overline{\beta}_i \gamma_{i,(j-1)/2} & n-2^l < i < 2^l \text{ and } j \text{ odd}; \\ 0 & \text{otherwise.} \end{cases}$$

Further, $\overline{\beta}_i \mapsto 0$ *if* $i \le \min(2^l-1, n-2^l)$.

Proof. By Lemma 1, $\overline{\beta}_{2j+1} = v_l^{-1}\overline{\varepsilon}_{j+2^{l-1}}$ if $j + 2^{l-1} \le \min(m', 2^l-1)$. The latter is a boundary in the Petrie complex. By Proposition 2, $\overline{\beta}_{j'(i)}$ eventually bounds in $E_{**}^{*}(SO(2n+1), B(l))$ if $i \le m'$ and $j(i) < n$. Thus $\overline{\beta}_j = 0$ in the E^{∞}-term if $j \le \min(2^l-1, n-2^l)$.

By Proposition 2.2, $\gamma_j(\overline{\beta}_i)$ corresponds to $\gamma_{i,j/2}$ in the Petrie complex if j is even. If j is odd, then $\gamma_j(\overline{\beta}_i) = \overline{\beta}_i\gamma_{i,(j-1)/2}$. Combining all this with the fact that the Petrie complex respects inclusions proves the lemma.

Proof of Theorem 1.1. Using Lemma 3, we see that the descriptions of $E_{**}^{\infty}(SO(2n+1), B(l))$ as an algebra given in Proposition 2 and in Theorem 1.1 agree. Further, it is generated by the images of the suspension

$$B(l)_*(\Omega SO(2n+1)) \to E_{1*}^{\infty}(SO(2n+1), B(l))$$

and of

$$E_{**}^{\infty}(SO(2^{l+1}-1), B(l)) \xrightarrow{incl._*} E_{**}^{\infty}(SO(2n+1), B(l)).$$

The former consists of coalgebra primitives. The effect of the diagonal on the latter can be determined using Proposition 2.2 and Lemma 3. The details are left to the reader.

Remarks. We can, at least in principle, calculate the effects of the stable BP-operations. To do this effectively however, we need to express $\{\overline{\beta}_{2i+1} \mid 2^{l-1} \le i \le m' - 2^{l-1}\}$ and $\{\overline{\alpha}_{2j} \mid m' < j < n\}$ in terms of the generators listed in Theorem 1.1. We can show, using the proofs of Lemmas 1 and 3, that $\overline{\beta}_{2i-2^l+1} = v_l^{-1}\sum_{i=2^l}^{2i+2} \nu_{i,2k-i}\overline{\beta}_i$ if $i \le m'$. However I have not been able to obtain an equally simple formula for the $\overline{\alpha}$'s due to integrality questions.

A similar remark applies to the Milnor primitives Q_i (see [JW] or [Wu] for a definition). In fact this is simpler as Q_i acts trivially on $B(l)_*(\Omega SO(2n+1))$. One can show that $Q_{l-1}\, P(l)_*(SO(2^{l+1}-1))$ is contained in the ideal generated by $\{\overline{\beta}_i \mid i < 2^l\}$. It now follows from Lemma 3 that Q_{l-1} acts trivially on $B(l)_*(SO(2n+1))$ if $n \geq 2^{l+1}$. By the results of [Wu], $B(l)_*(SO(2n+1))$ is then a cocommutative Hopf algebra and $B(l)^*(SO(2n+1))$ is the dual Hopf algebra. It can be shown that, if $n \geq 2^{l+1}$, then there is an extension

$$\bigotimes_{i=2^{l-1}}^{m'-2^{l-1}} E(\overline{\beta}_{2i}) \otimes \bigotimes_{i=2m'-2^l+2}^{n-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=m}^{n-1} E(\overline{\alpha}_{2i+1})$$

$$\longrightarrow B(l)_*(SO(2n+1)) \xrightarrow{g} \bigotimes_{i=0}^{2^{l-1}-1} \Gamma_{k(i)}(\gamma_i)$$

of Hopf algebras, with g a split epimorphism.

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