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HIGHER p-TORSION IN THE β-FAMILY

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ABSTRACT. We prove the existence of new families of v_2-periodic elements of the stable homotopy of the sphere detected in the second filtration of the Adams-Novikov Spectral Sequence for primes greater than 3. Our main corollary is that the p-component of π_*^s contains any finite abelian p-group as a subgroup in some dimension (for $p \geq 5$).

INTRODUCTION

The goal of this note is to provide a proof of a theorem giving elements of the p-component of π_*^s detected in the second filtration of the Adams-Novikov spectral sequence having arbitrarily high p-order, for $p \geq 5$. The main theorem is the following:

Theorem 1. *For* $j, k \geq 1$, $s \geq 2$, *and* $n > \log_2(jp^{2^{k+1}-2})$, *there is an element of* π_*^s *detected in the Adams-Novikov* E_2 *term by*

$$\beta_{sp^{2^{k+1}-1+n}/jp^{2^{k+1}-1}, 2^{k+1}} + p^{2^k}(\textit{sum of } \beta\textit{'s involving smaller powers of } v_2).$$

As a direct corollary of Theorem 1 we have

Corollary 1. *There are elements of arbitrarily high p-order in* π_*^s *detected in the second filtration of the Adams-Novikov spectral sequence (and hence in the cokernel of the J-homomorphism).*

Of course it is already known that the image of the J-homomorphism contains elements of arbitrarily high order, but we believe these are the first examples of such elements in the cokernel of the J-homomorphism. In §2 we also prove the following corollary, which was conjectured by Oka in [4].

Corollary 2. *The p-component of* π_*^s *(in fact the cokernel of the J-homomorphism) contains any finite abelian p-group as a subgroup (although not necessarily a direct summand) in some dimension.*

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In §1 we will establish some notation and some preliminary results. In §2 we will establish a result on the divisibility of certain homotopy classes that will imply Theorem 1, and we will also prove the corollaries above.

We have recently discovered that Lin Jinkun [1] has proven results stronger than Theorem 1 using different methods in work not yet published.

1. Definitions and lemmas

To simplify notation, we will define $\sigma(m,n)$ to be $m^{2^{n+1}-1} = m^{1+2+4+\cdots+2^n}$. Theorem 1 then asserts that $\beta_{sp^n\sigma(p,k)/j\sigma(p,k),2^{k+1}}$ + (error term), with the given restrictions on j,k,s and n, is a permanent cycle in the Adams-Novikov spectral sequence. For future use note that $\sigma(m,1) = m^3$ and $\sigma(m,n) = m\sigma(m^2,n-1)$.

The letter t will always stand for an integer not divisible by 2 or 3, and p will always be a prime greater than or equal to 5. The mod t Moore spectrum will be denoted by $M(t)$. The Bockstein for $M(t)$ will be denoted by $\delta_t\colon M(t) \to S^1 \to \Sigma M(t)$ (the two maps are projections to the top cell of $M(t)$ and inclusion of the bottom cell of $M(t)$).

A *split ring spectrum* K *over* $M(t)$ is a cofibre of a self map of dimension k of $M(t)$ such that $K\wedge K$ is homotopic to $K\vee \Sigma K \vee \Sigma^{k+1}K \vee \Sigma^{k+2}K$. The Bockstein for K will be denoted by $\delta'\colon K \to \Sigma^{k+1}M(t) \to \Sigma^{k+1}K$ (where the two maps are projections to the top Moore spectrum and inclusion of the bottom Moore spectrum), or by δ'_K if there is more than one split ring spectrum around. In [3, Lemma 2.6] it is shown that there is an extension of δ_t to a self map of K satisfying certain conditions. We will denote such an extension by δ (or δ_K if there is room for confusion).

For K a split ring spectrum, f in $[K,K]_r$, use the splitting of $K\wedge K$ to define a function $d\colon [K,K]_r \to [K,K]_{r+1}$ by

$$d(f)\colon \Sigma^{r+1}K \to \Sigma^r K\wedge K \xrightarrow{1\wedge f} K\wedge K \to K$$

and $d'\colon [K,K]_r \to [K,K]_{k+r+1}$ by

$$d'(f)\colon \Sigma^{k+r+1}K \to \Sigma^r K\wedge K \xrightarrow{1\wedge f} K\wedge K \to K.$$

Oka shows $d(\delta') = 0$ [3, equation 2.4], and $d(\delta) = -1$ [3, Lemma 2.6]. He also shows the splitting of $K\wedge K$ can be taken so that $d'(\delta) = 0$ [3, Lemma 4.4], and $d'(\delta') = -1$ [3, Lemma 4.5]. The functions d and d' also satisfy derivation-like properties; if f is in $[K,K]_m$ and g is in $[K,K]_n$

$$d(gf) = (-1)^m d(g)f + gd(f)$$

[3, Proposition 1.1] and

$$d'(gf) = (-1)^m d'(g)f + gd'(f) + D(g,f),$$

where $D(g,f) = 0$ if $d(g)$ or $d(f)$ is null [3, Corollary 5.2].

If R is a ring spectrum, $\mathrm{Mod}(R)$ will mean the subring of R-module self-maps of R. It is routine to show that if R is a commutative ring spectrum, then $\mathrm{Mod}(R)$ is a commutative subring of $[R,R]_*$. Another routine verification is that if K is a split ring spectrum, f is in $\mathrm{Mod}(K)$ if and only if $d(f) = d'(f) = 0$.

The rest of this section will be used to prove Proposition 2. That proposition shows that certain elements of π_*^s that factor through a split ring spectrum K over $M(t)$ are multiples by t of elements of π_*^S that factor through some split ring spectrum L over $M(t^2)$. We need L to be a split ring spectrum so we can use Proposition 2 inductively. It would be straightforward to verify Proposition 2 if we didn't require L to be a split ring spectrum; the goal of this section is Lemma 3, which is used in the proof of Proposition 2 to show that L can be taken to be a split ring spectrum over $M(t^2)$.

The following lemma follows from looking at the appropriate mapping cones.

Lemma 1. *Given a map of cofibre sequences*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \to & C(f) & \to & \Sigma A \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \downarrow \Sigma h_1 \\ C & \xrightarrow{g} & D & \to & C(g) & \to & \Sigma C \end{array}$$

so that all the squares commute strictly, a map k exists so that all rows and columns of the diagram below are cofibre sequences and all squares commute strictly.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \to & C(f) & \to & \Sigma A \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \downarrow \Sigma h_1 \\ C & \xrightarrow{g} & D & \to & C(g) & \to & \Sigma C \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C(h_1) & \xrightarrow{k} & C(h_2) & \to & C(h_3) & \to & \Sigma C(h_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma A & \xrightarrow{\Sigma f} & \Sigma B & \to & \Sigma C(f) & \to & \Sigma^2 A \end{array}$$

Proposition 1. *Let K be $M(t)$ or a split ring spectrum over $M(t)$. If f is in* $\mathrm{Mod}(K)$ *then f^t is in the center of* $[K,K]$.

Proof. In the case where K is a split ring spectrum, the ring $[K,K]$ has the direct sum decomposition [3, Theorem 5.5],

$$[K,K] = \mathrm{Mod}(K) \oplus \mathrm{Mod}(K)\delta \oplus \mathrm{Mod}(K)\delta' \oplus \mathrm{Mod}(K)\delta\delta'.$$

Since $\mathrm{Mod}(K)$ is a commutative subring of $[K,K]$, a map g in $\mathrm{Mod}(K)$ is in the center of $[K,K]$ if and only if it commutes with δ and δ'. The proposition then follows for split ring spectra from Lemma 2. If K is $M(t)$ then the result follows from the similar, simpler direct sum decomposition of $M(t)$ (see [3, Proposition 1.3]) and the relation $[f^t, \delta] = 0$. □

Lemma 2 [3, Corollary 5.7]. *If K is a split ring spectrum over $M(t)$ and $f \in \mathrm{Mod}(K)$, then f^t commutes with δ and δ'.*

Proof. We will prove that f^t commutes with δ; the proof that f^t commutes with δ' is exactly analogous. Without loss of generality, the dimension of f is even, since otherwise f^2 is null. Using the fact that d' is almost a derivation, $d'(f) = d'(\delta) = 0$ implies $d'(f^s\delta - \delta f^s) = 0$. Also, $d(f) = 0$ and $d(\delta) = -1_K$, so

$$d(f^s\delta - \delta f^s) = (-1)d(f^s)\delta + f^s d(\delta) - d(\delta)f^s - \delta d(f^s) = f^s(-1_K) - (-1_K)f^s = 0.$$

Therefore, $f^s\delta - \delta f^s$ is in $\mathrm{Mod}(K)$, so f commutes with $f^s\delta - \delta f^s$. The next step is to show inductively that $s | (f^s\delta - \delta f^s)$. We claim that

$$f^s\delta - \delta f^s = s(f^s\delta - f^{s-1}\delta f).$$

Suppose this formula holds for some $s \geq 1$. Then

$$0 = f(f^s\delta - \delta f^s) - (f^s\delta - \delta f^s)f = f^{s+1}\delta - f\delta f^s - f^s\delta f + \delta f^{s+1},$$

so

$$\begin{aligned} f^{s+1}\delta - \delta f^{s+1} &= f\delta f^s - \delta f^{s+1} + f^s\delta f - \delta f^{s+1} \\ &= (f\delta - \delta f)f^s + (f^s\delta - \delta f^s)f \\ &= f^s(f\delta - \delta f) + f[s(f^s\delta - f^{s-1}\delta f)] \\ &= (s+1)(f^{s+1}\delta - f^s\delta f). \end{aligned}$$

But since t is odd and K is a module spectrum over $M(t)$, 1_K has order t, so $[K, K]$ is a $\mathbf{Z}/t\mathbf{Z}$ module. Therefore,

$$f^t\delta - \delta f^t = t(f^t\delta - f^{t-1}\delta f) = 0. \quad \square$$

We need Lemma 3 to show that, in certain circumstances, a split ring spectrum over $M(t^2)$ can be manufactured as the cofibre of a self map of a split ring spectrum over $M(t)$. This lemma will be used in the next section to prove Proposition 2.

Lemma 3. *If* $a \in \mathrm{Mod}(M(t))$ *then, for any* $u \geq 1$, *there is a map* $b \in \mathrm{Mod}(M(t^2))$ *and a map* $\Delta \in [K, K]_{-1}$ *so that the rows and columns of the diagram below are all cofibre sequences and all squares commute up to homotopy.*

$$\begin{array}{ccccccc}
\Sigma^{t^3un-1}M(t) & \xrightarrow{\delta_t} & \Sigma^{t^3un}M(t) & \rightarrow & \Sigma^{t^3un}M(t^2) & \xrightarrow{\rho_{2,1}} & \Sigma^{t^3un}M(t) \\
\downarrow a^{t^3u} & & \downarrow a^{t^3}u & & \downarrow b^{t^2}u & & \downarrow a^{t^3}u \\
\Sigma^{-1}M(t) & \xrightarrow{\delta_t} & M(t) & \rightarrow & M(t^2) & \xrightarrow{\rho_{2,1}} & M(t) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}K & \xrightarrow{\Delta} & K & \rightarrow & L & \rightarrow & K \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{t^3un}M(t) & \xrightarrow{\delta_t} & \Sigma^{t^3un+1}M(t) & \rightarrow & \Sigma^{t^3un+1}M(t^2) & \xrightarrow{\rho_{2,1}} & \Sigma^{t^3un+1}M(t)
\end{array}$$

Proof. If $|a|$ is odd, a^2 is null, so $b = 0$ and $\Delta = \delta_t \vee \delta_t$ will work. Therefore we can assume $|a|$ is even. Since a^t is in the center of $[M(t), M(t)]$, the following diagram homotopy commutes:

$$\begin{array}{ccc} \Sigma^{tn-1}M(t) & \xrightarrow{\delta_t} & \Sigma^{tn}M(t) \\ \downarrow a^t & & \downarrow a^t \\ \Sigma^{-1}M(t) & \xrightarrow{\delta_t} & M(t) \end{array}$$

If we replace the $M(t)$'s on the right-hand side with $\mathrm{Cyl}(\delta_t)$'s (the mapping cylinder on δ_t), and the map δ_t with inclusion into the end of the mapping cylinder, then we will have replaced the horizontal maps with cofibrations and can then replace the right-hand vertical map induced by a^t with a homotopic map a_t' so that

$$\begin{array}{ccc} \Sigma^{tn-1}M(t) & \to & \Sigma^{tn}\mathrm{Cyl}(\delta_t) \\ \downarrow a^t & & \downarrow a_t' \\ \Sigma^{-1}M(t) & \to & \mathrm{Cyl}(\delta_t) \end{array}$$

commutes strictly. Then there is a map

$$b': C(\delta_t) \to C(\delta_t)$$

induced by a^t and a_t' so that

$$\begin{array}{ccccccc} \Sigma^{tn-1}M(t) & \to & \Sigma^{tn}\mathrm{Cyl}(\delta_t) & \to & \Sigma^{tn}C(\delta_t) & \xrightarrow{\rho_{2,1}} & \Sigma^{tn}M(t) \\ \downarrow a^t & & \downarrow a_t' & & \downarrow b' & & \downarrow a^t \\ \Sigma^{-1}M(t) & \to & \mathrm{Cyl}(\delta_t) & \to & C(\delta_t) & \xrightarrow{\rho_{2,1}} & M(t) \end{array}$$

commutes strictly. Therefore, by raising every vertical map to the power t^2u, we see that

$$\begin{array}{ccccccc} \Sigma^{t^3un-1}M(t) & \to & \Sigma^{t^3un}\mathrm{Cyl}(\delta_t) & \to & \Sigma^{t^3un}C(\delta_t) & \xrightarrow{\rho_{2,1}} & \Sigma^{t^3un}M(t) \\ \downarrow a^{t^3u} & & \downarrow a_t'^{t^2u} & & \downarrow b'^{t^2u} & & \downarrow a^{t^3u} \\ \Sigma^{-1}M(t) & \to & \mathrm{Cyl}(\delta_t) & \to & C(\delta_t) & \xrightarrow{\rho_{2,1}} & M(t) \end{array}$$

commutes strictly. Now, if we apply Lemma 1 to the above diagram and then replace $\mathrm{Cyl}(\delta_t)$ with the homotopy equivalent spectrum $M(t)$, and $C(\delta_t)$ with the homotopy equivalent spectrum $M(t^2)$, we get a homotopy commutative diagram as in the statement of Lemma 3 except that $b' \in [M(t^2), M(t^2)]$ is not necessarily in $\mathrm{Mod}(M(t^2))$. I claim that, in fact, b'^{t^2} is homotopic to b^{t^2} for some $b \in \mathrm{Mod}(M(t^2))$. The lemma then follows.

To verify this claim, first note that, by [3] Proposition 1.3, $b' = b + \delta_{t^2}c$ for $b, c \in \mathrm{Mod}(M(t^2))$. By the proof of Theorem 1.5 in [3] the next diagram (where b' has been replaced by b)

$$\begin{array}{ccccccc} \Sigma^{tn-1}M(t) & \xrightarrow{\delta_t} & \Sigma^{tn}M(t) & \to & \Sigma^{tn}M(t^2) & \xrightarrow{\rho_{2,1}} & \Sigma^{tn}M(t) \\ \downarrow a^t & & \downarrow a_t' & & \downarrow b & & \downarrow a^t \\ \Sigma^{-1}M(t) & \xrightarrow{\delta_t} & M(t) & \to & M(t^2) & \xrightarrow{\rho_{2,1}} & M(t) \end{array}$$

commutes up to homotopy. Therefore

$$a^t\rho_{2,1} = \rho_{2,1}b = \rho_{2,1}b' = \rho_{2,1}b + \rho_{2,1}\delta_{t^2}c$$

and so $\rho_{2,1}\delta_{t^2}c = 0$. Also, it is clear from the definitions that $\delta_{2,1}\delta_{t^2} = 0$ where $\delta_{2,1}$ is the Bockstein in $[M(t^2), \Sigma M(t)]$. Now $[M(t^2), \Sigma M(t)]$ is generated by $\delta_{2,1}$ and $[M(t^2), M(t)]$ is generated by $\rho_{2,1}$, so, since

$$M(t^2) \xrightarrow{t} M(t^2) \to M(t) \vee \Sigma M(t)$$

is a cofibre sequence, $t | \delta_{t^2}c$. Therefore $t^2 | (\delta_{t^2}c)^2$, so $(\delta_{t^2}c)^2 = 0$. It follows that

$$b'^{t^2} = (b + \delta_{t^2}c)^{t^2} = b^{t^2} + \delta_{t^2}cb^{t^2-1} + b\delta_{t^2}cb^{t^2-2} + \cdots + b^{t^2-1}\delta_{t^2}c.$$

Now b and c commute since they are in $\mathrm{Mod}(M(t^2))$. Let $\lambda = b\delta_{t^2} - \delta_{t^2}b$. Then, by the proof of Lemma 2, λ is in $\mathrm{Mod}(M(t^2))$, so it commutes with b and c. Since $\delta_{t^2}^2$ is null, λ commutes with δ_{t^2}. Therefore

$$(b + \delta_{M(t^2)}c)^{t^2} = b^{t^2} + t^2 \cdot \delta_{M(t^2)}cb^{t^2-1} + \frac{(t^2-1)\cdot t^2}{2}\lambda c b^{t^2-2} = b^{t^2}$$

up to homotopy, and so

$$b'^{t^2} = b^{t^2}$$

up to homotopy. That establishes the claim and hence the lemma. □

2. Divisibility of betas

The next proposition is the main element in the proof of Lemma 4, which is both the inductive step and the base case of Theorem 1. The rest of the proof of Lemma 4 is a matter of keeping track of BP homology.

Proposition 2. *Let $a \in \mathrm{Mod}(M(t))$ have dimension n, and $K = C(a^{u\cdot t^3})$ be a split ring spectrum over $M(t)$. Suppose f in $\mathrm{Mod}(K)$ has dimension v. Then there is a split ring spectrum $L = C(b^{u\cdot t^2})$ (where $b \in \mathrm{Mod}(M(t^2))$) over $M(t^2)$ and a map g in $\mathrm{Mod}(L)$ so that the following diagram commutes*:

$$\begin{array}{ccccccccc}
S^{tv} & \xrightarrow{i_K} & \Sigma^{tv}K & \xrightarrow{f^t} & K & \xrightarrow{\pi'_K} & \Sigma^{u\cdot t^3 n+1}M(t) & \xrightarrow{\pi_M} & S^{u\cdot t^3 n+2} \\
\downarrow \cdot t & & & & & & & & \downarrow 1_S \\
S^{tv} & \xrightarrow{i_L} & \Sigma^{tv}L & \xrightarrow{g} & L & \xrightarrow{\pi'_L} & \Sigma^{u\cdot t^3 n+1}M(t^2) & \xrightarrow{\pi_M} & S^{u\cdot t^3 n+2}
\end{array}$$

Proof. By Lemma 3, there is a spectrum L, cofibre of $b^{u\cdot t^2}$ for some $b \in \mathrm{Mod}(M(t^2))$ and cofibre of a map Δ in $[K, K]_{-1}$. Since $b \in \mathrm{Mod}(M(t^2))$, L is a split ring spectrum over $M(t^2)$ by [3, Proposition 2.9].

By Proposition 1, f^t commutes with Δ, so there is a map g' in $[L, L]$ making the following diagram commute:

$$\begin{array}{ccccccc}
\Sigma^{tv-1}K & \xrightarrow{\Delta} & \Sigma^{tv}K & \to & \Sigma^{tv}L & \to & \Sigma^{tv}K \\
\downarrow f^t & & \downarrow f^t & & \downarrow g' & & \downarrow f^t \\
\Sigma^{-1}K & \xrightarrow{\Delta} & K & \to & L & \to & K
\end{array}$$

This shows that the following diagram commutes:

$$\begin{array}{ccccccccc} S^{tv} & \xrightarrow{i_K} & \Sigma^{tv}K & \xrightarrow{f^t} & K & \xrightarrow{\pi'_K} & \Sigma^{u\cdot t^3n+1}M(t) & \xrightarrow{\pi_M} & S^{u\cdot t^3n+2} \\ \downarrow \cdot t & & & & & & & & \downarrow 1_S \\ S^{tv} & \xrightarrow{i_L} & \Sigma^{tv}L & \xrightarrow{g'} & L & \xrightarrow{\pi'_L} & \Sigma^{u\cdot t^3n+1}M(t^2) & \xrightarrow{\pi_M} & S^{u\cdot t^3n+2} \end{array}$$

The decomposition of $[L,L]$ shows that g' can be expressed as a sum $g + g_1\delta_L + g_2\delta'_L + g_3\delta_L\delta'_L$ with $g, g_i \in \mathrm{Mod}(L)$. But $\delta'_L i'_L = 0$, where i'_L is inclusion of the bottom Moore spectrum, since δ'_L is the Bockstein for L. Also,

$$\delta_L i_L = \delta_L i'_L i_{M(t^2)} = i'_L \delta_{t^2} i_{M(t^2)} = 0$$

by construction of δ_L. So $g' i_L = g i_L$ and the proposition follows. □

Corollary 3. *With the hypotheses above, t divides $\pi_K f^t i_K$.*

Now let $K = C(a^{u\cdot p^{3i}})$ be a split ring spectrum over $M(p^i)$, where $a \in \mathrm{Mod}(M(p^i))$ has the effect on $\mathrm{BP}_* M(p^i)$ of multiplication by v_1^j.

Lemma 4. *Suppose $f \in \mathrm{Mod}(K)$ is such that*

$$\mathrm{BP}_*(f) = v_2^t + p^{i'}(\textit{some polynomial in } v_1 \textit{ and } v_2),$$

where $i' \geq i/2$. Then p^i divides a homotopy element detected by $\beta_{tp^i/ju\cdot p^{3i},i}$. Also there is a split ring spectrum $L = C(b^{u\cdot p^{2i}})$ for some $b \in \mathrm{Mod}(M(p^{2i}))$ over $M(p^{2i})$ and a map $g \in \mathrm{Mod}(L)$ such that

$$\mathrm{BP}_*(g) = v_2^{tp^i} + p^i\ (\textit{some polynomial in } v_1 \textit{ and } v_2)$$

and $p^i \pi_L g i_L = \pi_K f^{p^i} i_K$.

Proof. By Proposition 2 there is a spectrum L as described and a map $g_0 \in \mathrm{Mod}(L)$ such that

$$\pi_K f^{p^i} i_K = p^i \pi_L g_0 i_L.$$

Since

$$\mathrm{BP}_*(f) = v_2^t + p^{i'}(\text{polynomial with only lower powers of } v_2)$$

it follows that

$$\mathrm{BP}_*(f^{p^i}) = v_2^{tp^i},$$

so $\pi_K f^{p^i} i_K$ is detected by $\beta_{tp^i/ju\cdot p^{3i},i}$ (see [2]). This establishes the first claim. Now, from the proof of Proposition 2,

$$\begin{aligned} \mathrm{BP}_*(g_0) &= \mathrm{BP}_*(g') = \mathrm{BP}_*(f^{p^i}) + p^i(\text{something}) \\ &= (1 + ap^i)v_2^{tp^i} + p^i(\text{polynomial with only lower powers of } v_2) \end{aligned}$$

for some a. But $1+ap^i$ is a unit $\bmod p^{2i}$, so, since 1_L has order p^{2i}, we can compose g_0 with a self equivalence of L inducing multiplication by $1-ap^i$. Call the resulting map g. Then

$$\mathrm{BP}_*(g) = v_2^{tp^i} + p^i(\text{polynomial with only lower powers of } v_2)$$

and, since $p^i(1-ap^i) = p^i$ on L,

$$\pi_K f^{p^i} i_K = p^i \pi_L g i_L. \quad \square$$

Proposition 3. *Let $j \geq 1, k \geq 0$. Suppose $K = C(a^{\sigma(p^i,k)})$ is a split ring spectrum over $M(p^i)$ and $f \in \mathrm{Mod}(K)$ is such that*

$$\mathrm{BP}_*(f) \equiv v_2^t \mod p^{i'} \qquad \text{where } i' \geq i/2.$$

Then $p^{2^k} | \pi_K f^{\sigma(p^i,k-1)} i_K$, which is detected by $\beta_{t\cdot\sigma(p^i,k-1)/j\cdot\sigma(p^i,k),i}$, and there is an element x of π_*^S detected by $\beta_{t\cdot\sigma(p^i,k-1)/j\cdot\sigma(p^i,k),2^k i} + p^{2^{k-1}i}$($\beta$'s involving lower powers of v_2). Furthermore, $x = \pi_L g i_L$, where g is in $[L, L]$ for some split ring spectrum $L = C(b^{p^{2^k i}})$ with $b \in \mathrm{Mod}(M(p^{2^k i}))$.

Proof. The proof is by induction on k. For $k = 0$ the result is trivial (for $k = 1$ the proposition reduces to Lemma 4). Now suppose the proposition holds for some $k - 1 \geq 0$ and that we have f and K satisfying the hypotheses for $\beta_{t/j\cdot\sigma(p^i,k),i}$. Then, by Lemma 4, there is a split ring spectrum $\overline{K} = C(\overline{a}^{\sigma(p^{2i},k-1)})$ over $M(p^{2i})$ and a map $\overline{f} \in \mathrm{Mod}(\overline{K})$ so that $p^i \pi_{\overline{K}} \overline{f} i_{\overline{K}} = \pi_K f^{p^i} i_K$ and $\mathrm{BP}_*(\overline{f})$ is as in Lemma 4. But now, by the inductive hypotheses, there is a split ring spectrum $L = C(b^{p^{2^{k-1}2i}})$ and a map $g \in \mathrm{Mod}(L)$ so that $\pi_L g i_L$ is detected by

$$\beta_{tp^i\cdot\sigma(p^{2i},k-2)/j\cdot\sigma(p^i,k),2^{k-1}2i} = \beta_{t\sigma(p^i,k-1)/j\cdot\sigma(p^i,k),2^k i} \qquad (\mathrm{mod}\, p^{2^{k-1}i}) \quad \square$$

Corollary 4. *In the Adams-Novikov E_2 term, for $j, k \geq 1, n > \log_2(\frac{j\sigma(p,k)}{p})$, and $s \geq 2$, $\beta_{sp^n\sigma(p,k-1)/j\cdot\sigma(p,k),2^k} + p^{2^{k-1}}$ (lower order terms) detects an element of homotopy that can be written as $\pi_L f i_L$ for some split ring spectrum K over $M(p^{2^k})$ where $f \in \mathrm{Mod}(L)$.*

Proof. By [3, Theorem I], the hypotheses of Proposition 3 are satisfied for split ring spectra K with $\mathrm{BP}_*(K) = \mathrm{BP}_*/(p, v_1^{j\cdot\sigma(p,k)})$, $f \in \mathrm{Mod}(K)$ with $\mathrm{BP}_*(f) = v_2^{sp^n}$. $\square$

Proof of Theorem 1. Note that, since $g^{p^{2^k}} \in \mathrm{Center}([L, L])$, we can produce a spectrum $\overline{L} = C(c) = C(\Delta_L)$ where $c \in \mathrm{Mod}(M(p^{2^k}))$ (see proof of Proposition 2) and $\Delta \in [L, L]_{-1}$ and a self map of $\overline{L}$, $\overline{g}$ so that $\pi_{\overline{L}} \overline{g} i_{\overline{L}}$ is detected by $\beta_{sp^n\sigma(p,k)/j\sigma(p,k),2^{k+1}} (\mathrm{mod}\, p^{2^k})$. By [2, Theorem 2.6], p does not divide $\beta_{sp^n\sigma(p,k)/j\sigma(p,k),2^{k+1}}$, so this is the highest order torsion element detected by a β that we can hope to get from $\beta_{sw/j\sigma(p,k)}$ for any w.

Proof of Corollary 1. The element detected in the proof immediately above—call it $b_{sp^n\sigma(p,k)/j\sigma(p,k),2^{k+1}}$—has order dividing $p^{2^{k+1}}$ since $p^{2^{k+1}}\cdot 1_{\overline{L}} = 0$ (because $\overline{L}$ is a module spectrum over $M(p^{2^{k+1}})$ since $c \in \mathrm{Mod}(M(p^{2^{k+1}}))$; so $p^{2^{k+1}}\cdot\overline{f} = 0$. But

$$\sigma(p,k)\beta_{sp^n\sigma(p,k)/j\sigma(p,k),2^{k+1}} = \beta_{sp^n\sigma(p,k)/j\sigma(p,k)},$$

so $b_{sp^n\sigma(p,k)/j\sigma(p,k),2^{k+1}}$ has order exactly $p^{2^{k+1}}$. □

Corollary 5. *If* $s \geq 2$, $r, j \geq 1$, $n \geq \log_2(jp^{r-2})$, *and* $k \leq \log_2(r+1)$, *then modulo elements of higher* BP*-filtration,* $p^{2^k-1} | b_{sp^{n+r}/jp^r}$, *where* b_{sp^{n+r}/jp^r} *is some element detected by* β_{sp^{n+r}/jp^r}.

Proof of Corollary 2. We can produce a subgroup of some stable homotopy group of the form $(\mathbf{Z}/p^{2^{k+1}})^i$ as follows. Define

$$\begin{aligned} j_0 &= 1, & n_0 &> \log_2(\sigma(p,k)), \\ j_1 &= 1 + p^{n_0}(p+1), & n_1 &> \log_2(\sigma(p,k)) + \log_2(j_1), \\ j_i &= j_{i-1} + p^{n_{i-1}}(p+1), & n_i &> \log_2(j_i) + \log_2(\sigma(p,k)). \end{aligned}$$

Then the classes $\beta_{(sp^{n_i} - p^{n_{i-1}} - \cdots p^{n_{i-j}})\sigma(p,k)/j_{i-j}\sigma(p,k),2^{k+1}}$, for $0 \leq j \leq i$, all are leading terms of infinite cycles in the Adams-Novikov spectral sequence detecting independent homotopy classes of order $p^{2^{k+1}}$, by Theorem 1. It is easy to check that they all live in $\pi^S_{sp^{n_i}\sigma(p,k)(p+1)q - j_i\sigma(p,k)q-2}$. □

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